2009

Problem 10

Centro-Symmetric Shadows

A set of points in the plane is said to be *centro-symmetric* if it has a centre of symmetry. For example, the vertices of a square form a centro-symmetric set. (Recall that a *centre of symmetry* of a set S in the plane is a point c with the property that for any point $p \in S$ there exists a point $p' \in S$ such that p and p' are equidistant from c and lie on a line passing thought c.) Given a set of points, its shadow on a line is its orthogonal projection onto this line.

1. Let n > 3 be a positive integer. Denote by k(n) the minimum positive integer k with the following property:

    for any set S of n points in the plane, if there exist k lines, no two parallel, such that for each line the shadow of S on this line is centro-symmetric, then the initial set S is also centro-symmetric. Find or estimate the number k(n).

2. Formulate and study 3-dimensional analogs of the problem.

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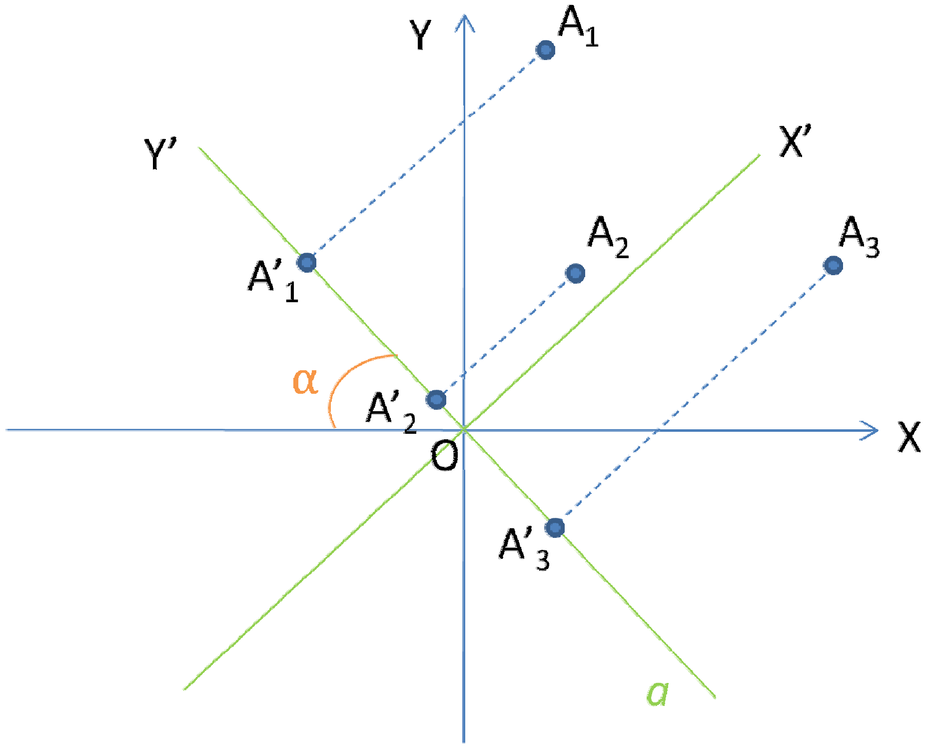
Find or estimate the number k(n).

I. Let us take three points with polar coordinates $A_1(r_1, \varphi_1)$, $A_2(r_2, \varphi_2)$ and $A_3(r_3, \varphi_3)$ and $\varphi_1, \varphi_2, \varphi_3 \in [0, 2\Pi)$. Let the shadow of A_1, A_2, A_3 on the line ***a*** be centro-symmetric



The shadow of A_1, A_2, A_3 is centro-symmetric on any line parallel to ***a***. Consequently the line ***a*** is defined only by the angle between the line ***a*** and the X-axis. Let this angle be $\alpha \in [0, \Pi)$. Without loss of generality we consider the lines up to **α**. $\sphericalangle XOX' = 90° - \alpha$, therefore:

$\sphericalangle X'OA_1 = \varphi_1 - 90° + \alpha \quad => \quad OA'_1 = \sin(\varphi_1 - 90° + \alpha).r_1 = \cos(\varphi_1 + \alpha).r_1$
$\sphericalangle X'OA_2 = \varphi_2 - 90° + \alpha \quad => \quad OA'_2 = \sin(\varphi_2 - 90° + \alpha).r_2 = \cos(\varphi_2 + \alpha).r_2$
$\sphericalangle X'OA_3 = \varphi_3 - 90° + \alpha \quad => \quad OA'_3 = \sin(\varphi_3 - 90° + \alpha).r_3 = \cos(\varphi_3 + \alpha).r_3$

But $A'_2 \in A'_1A'_3$ and $A'_1A'_2 = A'_2A'_3$, consequently:

$$2OA'_2 = OA'_1 + OA'_3$$

$$2\cos(\varphi_2 + \alpha).r_2 = \cos(\varphi_1 + \alpha).r_1 + \cos(\varphi_3 + \alpha).r_3$$

$$2\cos\varphi_2.\cos\alpha.r_2 - 2\sin\varphi_2.\sin\alpha.r_2 = \cos\varphi_1.\cos\alpha.r_1 - \sin\varphi_1.\sin\alpha.r_1 + \cos\varphi_3.\cos\alpha.r_3 - \sin\varphi_3.\sin\alpha.r_3$$

$$(2\cos\varphi_2.r_2 - \cos\varphi_1.r_1 - \cos\varphi_3.r_3).\cos\alpha = \sin\alpha.(2\sin\varphi_2.r_2 - \sin\varphi_1.r_1 - \sin\varphi_3.r_3)$$

1.) If $2\cos\varphi_2.r_2 - \cos\varphi_1.r_1 - \cos\varphi_3.r_3 = 0$ and $2\sin\varphi_2.r_2 - \sin\varphi_1.r_1 - \sin\varphi_3.r_3 = 0$, then $A_2 \in A_1A_3$ и $A_1A_2 = A_2A_3$. Consequently A_1, A_2, A_3 form a centro-symmetric set and $\alpha \in [0, \Pi)$.

Conclusion #1: For any three points, which form a centro-symmetric set S, there are an infinite number of lines, on which the shadow of S is also centro-symmetric.

2.) If $2\cos\varphi_2.r_2 - \cos\varphi_1.r_1 - \cos\varphi_3.r_3 = 0$ and $2\sin\varphi_2.r_2 - \sin\varphi_1.r_1 - \sin\varphi_3.r_3 \neq 0$, then $\sin\alpha = 0$. Consequently $\alpha = 0°$ and only one line exists, on which the shadow of the three points is centro-symmetric.

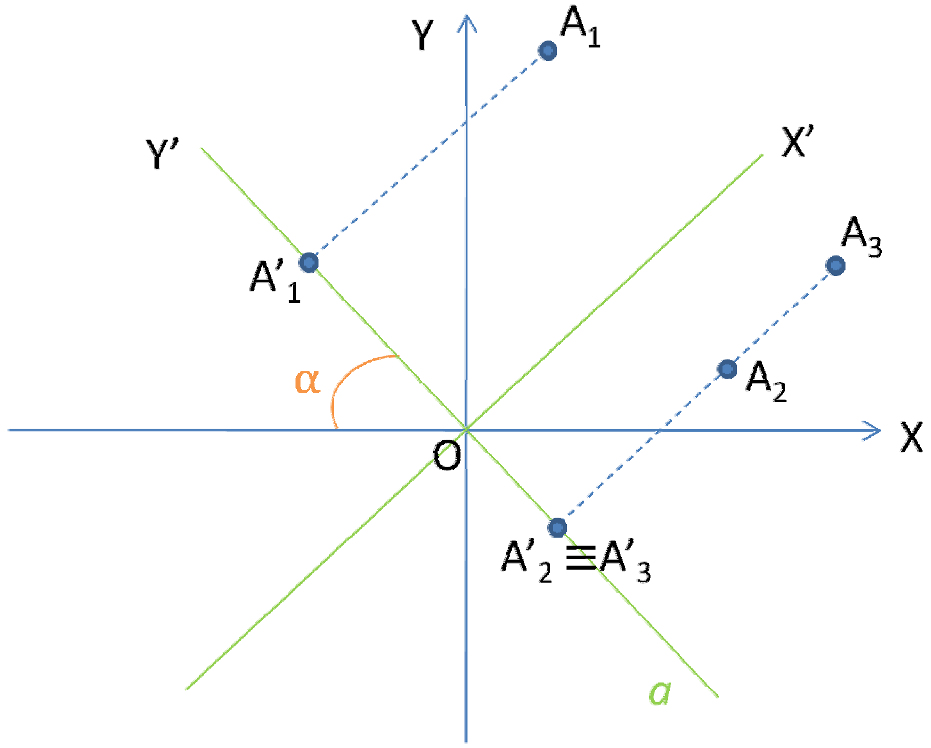
3.) If $2\cos\varphi_2.r_2 - \cos\varphi_1.r_1 - \cos\varphi_3.r_3 \neq 0$ and $2\sin\varphi_2.r_2 - \sin\varphi_1.r_1 - \sin\varphi_3.r_3 = 0$, then $\cos\alpha = 0$. Consequently $\alpha = 90°$ and only one line exists, on which the shadow of the three points is centro-symmetric.

4.) If $2\cos\varphi_2.r_2 - \cos\varphi_1.r_1 - \cos\varphi_3.r_3 \neq 0$ and $2\sin\varphi_2.r_2 - \sin\varphi_1.r_1 - \sin\varphi_3.r_3 \neq 0$, then

$$\tan\alpha = \frac{2\cos\varphi_2.r_2 - \cos\varphi_1.r_1 - \cos\varphi_3.r_3}{2\sin\varphi_2.r_2 - \sin\varphi_1.r_1 - \sin\varphi_3.r_3}$$

Consequently $\alpha = \tan^{-1}\left(\frac{2\cos\varphi_2.r_2 - \cos\varphi_1.r_1 - \cos\varphi_3.r_3}{2\sin\varphi_2.r_2 - \sin\varphi_1.r_1 - \sin\varphi_3.r_3}\right)$ and only one line exists, on which the shadow of the three points is centro-symmetric.

II. Let's take three points with polar coordinates $A_1(r_1, \varphi_1)$, $A_2(r_2, \varphi_2)$ and $A_3(r_3, \varphi_3)$ and $\varphi_1, \varphi_2, \varphi_3 \in [0, 2\Pi)$. Let the shadow of A_1, A_2, A_3 on the line ***a*** be centro-symmetric



$OA'_2 = OA'_3$ therefore:

$OA'_2 - OA'_3 = 0$

$\cos(\varphi_2 + \alpha).r_2 - \cos(\varphi_3 + \alpha).r_3 = 0$

$\cos\varphi_2.\cos\alpha.r_2 - \sin\varphi_2.\sin\alpha.r_2 - \cos\varphi_3.\cos\alpha.r_3 + \sin\varphi_3.\sin\alpha.r_3 = 0$

$(\cos\varphi_2.r_2 - \cos\varphi_3.r_3).\cos\alpha = \sin\alpha.(\sin\varphi_2.r_2 - \sin\varphi_3.r_3)$

1.) If $\cos \varphi_2.r_2 = \cos \varphi_3.r_3$ and $\sin \varphi_2.r_2 = \sin \varphi_3.r_3$, then $A_2 \equiv A_3$, which contradicts to our assumption that we have three different points.

2.) If $\cos \varphi_2.r_2 \neq \cos \varphi_3.r_3$ and $\sin \varphi_2.r_2 = \sin \varphi_3.r_3$, then $\cos\alpha = 0$.
Consequently $\alpha = 90°$ and only one line exists, on which the shadow of the three points is centro-symmetric.

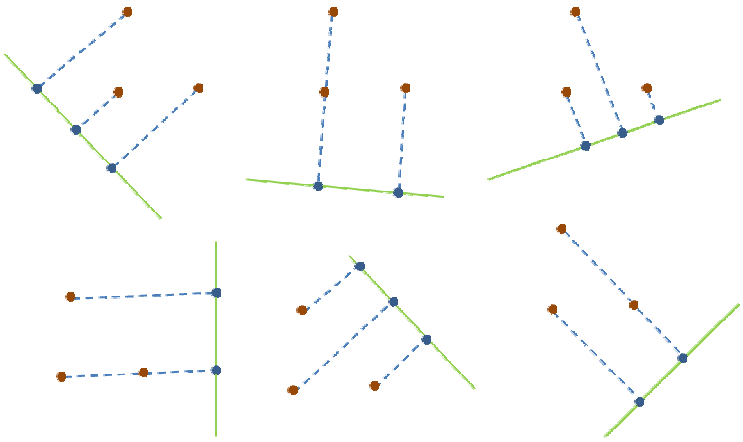
3.) If $\cos \varphi_2.r_2 = \cos \varphi_3.r_3$ and $\sin \varphi_2.r_2 \neq \sin \varphi_3.r_3$, then $\sin\alpha = 0$.
Consequently $\alpha = 0°$ and only one line exists, on which the shadow of the three points is centro-symmetric.

4.) If $\cos \varphi_2.r_2 \neq \cos \varphi_3.r_3$ and $\sin \varphi_2.r_2 \neq \sin \varphi_3.r_3$, then

$$\tan\alpha = \frac{\cos\varphi_2\,.r_2 - \cos\varphi_3.r_3}{\sin\varphi_2.r_2 - \sin\varphi_3.r_3}$$

Consequently $\alpha = \tan^{-1}\left(\frac{\cos\varphi_2.r_2 - \cos\varphi_3.r_3}{\sin\varphi_2.r_2 - \sin\varphi_3.r_3}\right)$ and only one line exists, on which the shadow of the three points is centro-symmetric.

The shadow of each of the three given points can be the centre of symmetry of the shadow of the points. The projection of any of the three points can coincide with the projection of one of the other two points. Therefore for any three points which are not centro-symmetric the maximum count of lines, on which their shadow is centro-symmetric, issix. If there are seven or more lines then the three points are centro-symmetric.



The figure illustrates the six lines that exist for every three points, which do not lie on a single line.

Conclusion #2: Three points are centro-symmetric if and only if there are at least seven lines, on which the shadow of the points is centro-symmetric. The minimum number of lines for 3 points is 7, so k(3) = 7.

Let's examine a set S of n points:

1. If n is an even number, then there are $\frac{n}{2}$ pairs of points, which should be centro-symmetric and the centre of symmetry is a point T which does not belong to S. Let's take one pair and the point T. These three points are centro-symmetric if and only if there are 7 lines, on which their shadow is centro-symmetric. So we increase the number of needed lines with 7 and decrease the number of the pairs examined in S with one. Consequently with finite number of steps $\frac{n}{2}$ we will be able to guarantee that S is centro-symmetric. Therefore the minimum number of lines is $k(n) = \frac{7n}{2}$.

2. If n is an odd number, then there are $\frac{n-1}{2}$ pairs of points, which should be centro-symmetric and the centre of symmetry is a point T which belongs to S. Let's take one pair and the point T. These three points are centro-symmetric if and only if there are 7 lines, on which their shadow is centro-symmetric. So we increase the

number of needed lines with 7 and decrease the number of the pairs examined in S with one. Consequently with finite number of steps $\frac{n-1}{2}$ we will be able to guarantee that S is centro symmetric. Therefore the minimum number of lines is $k(n) = \frac{7(n-1)}{2}$.

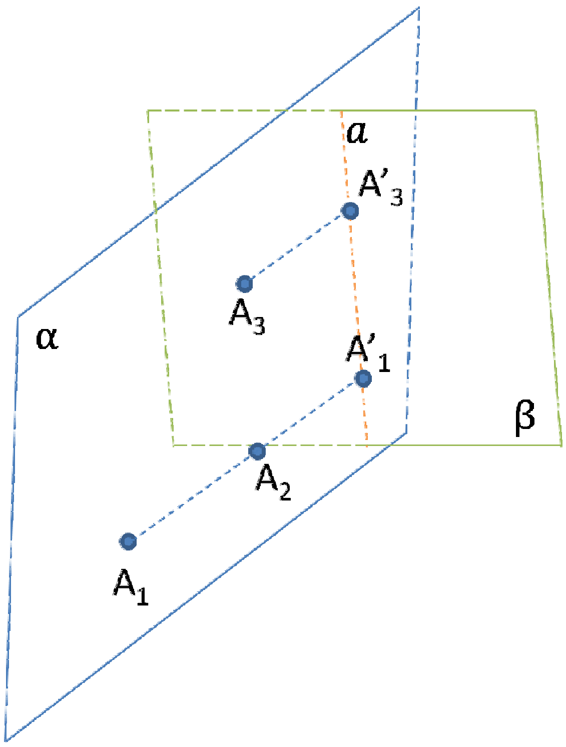
$$k(n) = \begin{cases} k(n) = \frac{\mathbf{7n}}{\mathbf{2}}, \text{ for n even} \\ k(n) = \frac{\mathbf{7(n-1)}}{\mathbf{2}}, \text{ for n odd} \end{cases}$$

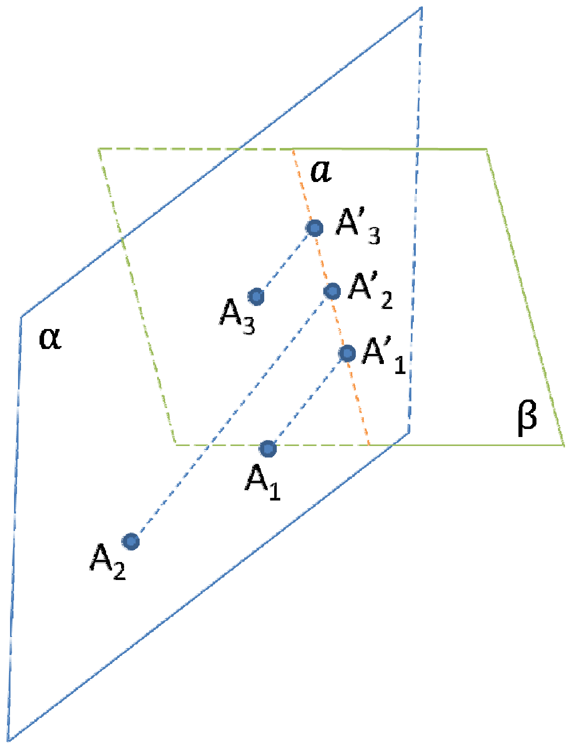
2. Formulate and study 3-dimensional analogs

Let $n > 3$ be a positive integer. Denote by d(n) the minimum positive integer d with the following property:

> for any set S of n points in the 3-dimensional space, if there exist d planes, no two parallel, such that for each plane the shadow of S on this plane is centro-symmetric, then the initial set S is also centro-symmetric. Find or estimate the number d(n).

Let us take tree points in the 3-dimensional space A_1, A_2, A_3 and $A_1, A_2, A_3 \in \alpha$ (if A_1, A_2 and A_3 belong to a single line then there are many planes α and if A_1, A_2 and A_3 do not belong to a single line there is a single plane α). Let us chose plane β and $\alpha \cap \beta = a$. Let the shadows of A_1, A_2, A_3 on β are A'_1, A'_2, A'_3. A'_1, A'_2, A'_3 form a centro-symmetric set; therefore the tree points are on a single line. A_1, A_2, A_3 are projected into line on β if and only if $\alpha \perp \beta$. Consequently $A'_1, A'_2, A'_3 \in a$. Let us examine the plane α. According to the results from the first part of the problem 7 lines, on which the shadow of A_1, A_2, A_3 is centro-symmetric, are needed to guarantee that A_1, A_2, A_3 is centro-symmetric set in α. Each of these lines defines a plane perpendicular to α, therefore seven planes, on which the shadows of A_1, A_2, A_3 are centro-symmetric, are needed to guarantee that A_1, A_2, A_3 form a centro-symmetric set in the 3-dimensional space.





Conclusion #3: For any three points in the 3-dimensional space, which form a centro-symmetric set S, there are an infinite number of planes, on which the shadow of S is also centro-symmetric.

Conclusion #4: Three points in the 3-dimensional space are centro-symmetric if and only if there are at least seven planes, on which the shadow of the points is centro-symmetric. The minimum number of planes for 3 points is 7, so d(3) = 7.

Let us examine a set S of n points in the 3-dimensional space:

1. If n is an even number, then there are $\frac{n}{2}$ pairs of points, which should be centro-symmetric and the centre of symmetry is a point T which does not belong to S. Let us take one pair and the point T. These three points are centro-symmetric if and only if there are 7 planes, on which their shadow is centro-symmetric. So we increase the number of needed planes with 7 and decrease the number of the pairs examined in S with one. Consequently with finite number of steps $\frac{n}{2}$ we will be able to guarantee that S is centro-symmetric. Therefore the minimum number of planes is $d(n) = \frac{7n}{2}$. On each of these planes, the maximum number of projected points is n, so $k_1(n) \leqq \left(\frac{7n}{2}\right)^2$, where $k_1(n)$ is the number of lines for a set S of n points in the 3-dimensional space needed to guarantee that S is centro-symmetric.

2. If n is an odd number, then there are $\frac{n-1}{2}$ pairs of points, which should be centro-symmetric and the centre of symmetry is a point T which belongs to S. Let's take one pair and the point T. These three points are centro-symmetric if and only if there are 7 planes, on which their shadow is centro-symmetric. So we increase the number of needed planes with 7 and decrease the number of the pairs examined in S with one. Consequently with finite number of steps $\frac{n-1}{2}$ we will be able to guarantee that S is centro-symmetric. Therefore the minimum number of planes is $d(n) = \frac{7(n-1)}{2}$. On each plane, the maximum number of projected points is n, so $k_1(n) \leqq \left(\frac{7(n-1)}{2}\right)^2$, where $k_1(n)$ is the number of lines for a set S of n points in the 3-dimensional space needed to guarantee that S is centro-symmetric.

$$d(n) = \begin{cases} d(n) = \dfrac{\mathbf{7n}}{\mathbf{2}}, \text{ for n even} \\ d(n) = \dfrac{\mathbf{7(n-1)}}{\mathbf{2}}, \text{ for n odd} \end{cases}$$

$$k_1(n) = \begin{cases} k_1(n) \leqq \left(\dfrac{\mathbf{7n}}{\mathbf{2}}\right)^{\mathbf{2}}, \text{ for n even} \\ k_1(n) \leqq \left(\dfrac{\mathbf{7(n-1)}}{\mathbf{2}}\right)^{\mathbf{2}}, \text{ for n odd} \end{cases}$$