2009

Problem 1

Specular Colourings

1. What is the maximum number of cells of an $m \times n$ grid that can be coloured blue, such that no two blue cells are symmetric with respect to any horizontal or vertical line of the grid?

2. Some cells of an $m \times n$ grid are coloured blue. We call such a colouring specular if for any interior horizontal or vertical line of the grid there are two blue cells that are symmetric with respect to this line. Denote by $S(m, n)$ the minimal number of blue cells in a specular colouring of an $m \times n$ grid.
Find $S(m, n)$ or estimate it (give lower and upper bounds).

3. Formulate and investigate 3-dimensional analogs of the problem.

Sofia High School of Mathematics

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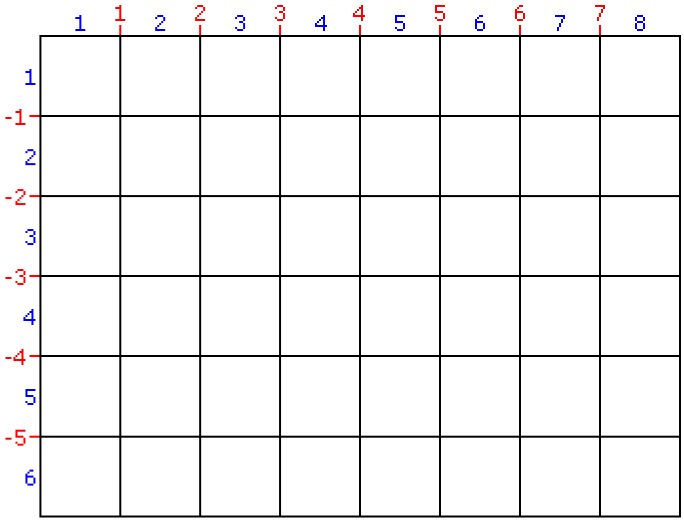
2 Problem 1

1. What is the maximum number of cells of an m × n grid that can be coloured blue, such that no two blue cells are symmetric with respect to any horizontal or vertical line of the grid?

Let us denote a grid m x n with G(m,n). Also let us use coordinate system for the cells of the grid with origin (1,1) in the top left corner, where the horizontal coordinates increase from left to right, while the vertical coordinates increase from top to bottom. We call the horizontal coordinate of a cell x, and the vertical y.

Let us also denote the lines in the grid. We denote the vertical lines with the numbers from 1 to m–1 from left to right, while the horizontal with the numbers from –1 to –n+1 from top to bottom for G(m,n).



G(8,6)

Let us examine the cells (x_1,y_1) and (x_2,y_2), symmetrical with respect to a vertical line v.

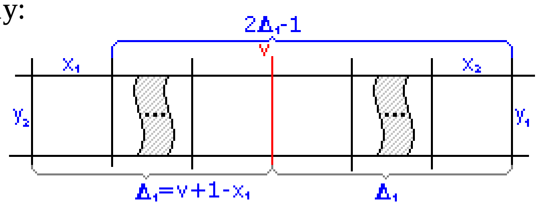
$x_2 = x_1 + 2\Delta_1 - 1$, where $\Delta_1 = v + 1 - x_1$. Consequently:

$x_2 = x_1 + 2(v + 1 - x_1) - 1$

$y_2 = y_1$

$x_2 = 2v - x_1 + 1$

$y_2 = y_1$



*Note: Δ_1 can be less than 0.

Conclusion #1: $x_2 - x_1 = 2\Delta_1 - 1$, which is an odd number, therefore x_1 and x_2 have different parity.

Let us examine the cells (x_1, y_1) and (x_2, y_2), symmetrical with respect to a horizontal line h.

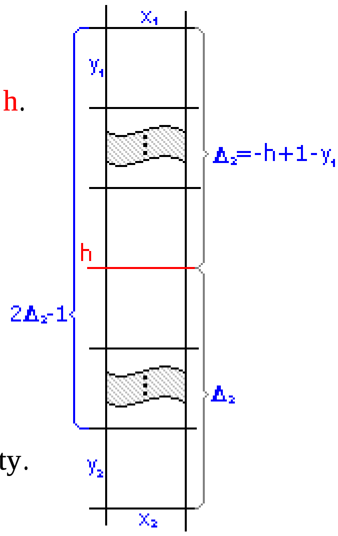
$y_2 = y_1 + 2\Delta_2 - 1$, where $\Delta_2 = -h + 1 - y_1$.

$y_2 = y_1 + 2(-h + 1 - y_1) - 1$

$x_2 = x_1$

$y_2 = -2h - y_1 + 1$

$x_2 = x_1$

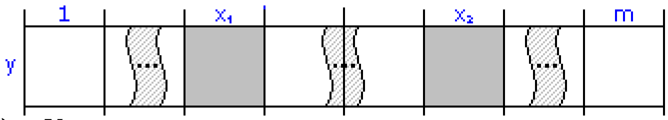


*Note: Δ_2 can be less than 0

Conclusion #2: $y_2 - y_1 = 2\Delta_2 - 1$, which is odd, therefore y_1 and y_2 have different parity.

3 Problem 1

1. • Let us examine a row y from $G(m,n)$. Let us choose two cells from the row with coordinates (x_1, y) and (x_2, y), where $x_1 = 2p$, while $x_2 = 2q - 1$; $p, q \in \mathbb{N}$; $x_1, x_2 \leq m$.



(x_1, y) and $(x_2; y)$ are symmetric if and only if

there is a single v, such that $x_2 = -x_1 + 2v + 1$; $v \in [1; m) \cap \mathbb{N}$;

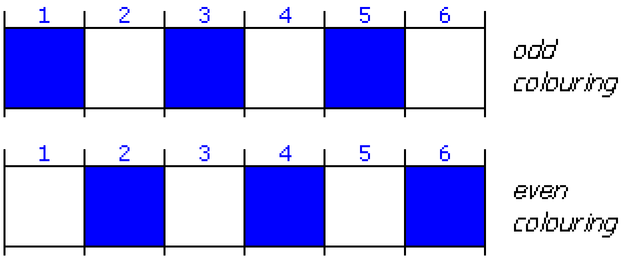
$$2v = x_1 + x_2 - 1$$

$$v = \frac{2p+2q-1-1}{2} = p + q - 1 \in \mathbb{N}$$

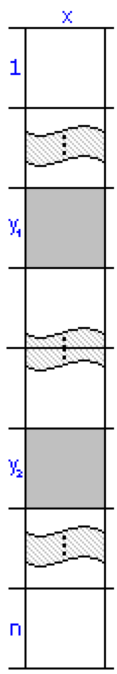
$$p + q - 1 \geq 1 + 1 - 1 = 1$$

$p + q - 1 \leq \frac{m}{2} + \frac{m+1}{2} - 1 = m - \frac{1}{2}$ therefore $p + q - 1 \in [1; m)$ and there is a single v.

$(x_1; y)$ and $(x_2; y)$ are symmetric for any x_1 and x_2 which have different parity. Consequently if there is a cell coloured in blue with odd x, there cannot be such with even x and vice versa. Therefore the maximum number of cells, which can be coloured in blue in a single row, is either all cells with even x or all cells with odd x.



, Let us examine a column x from $G(m, n)$ and choose two cells with coordinates (x, y_1) and (x, y_2), where $y_1 = 2p$, and $y_2 = 2q - 1$; $p, q \in \mathbb{N}$; $y_1, y_2 \leq n$. $(x; y_1)$ and $(x; y_2)$ are symmetric if and only if there exists a single h: $y_2 = -y_1 - 2h + 1$; $h \in [-n + 1, -1] \cap \mathbb{Z}$



$$-2h = y_1 + y_2 - 1$$

$$h = -\frac{2p+2q-1-1}{2} = -(p + q - 1) = 1 - p - q \in \mathbb{Z}$$

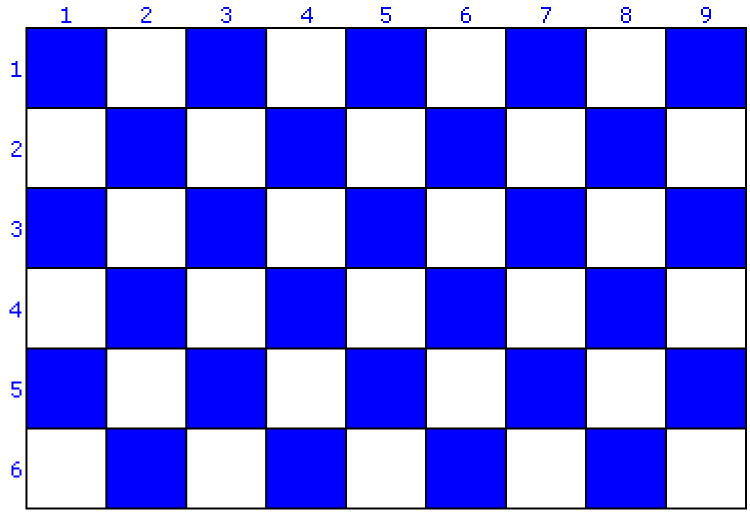
$$1 - p - q \leq 1 - 1 - 1 = -1$$

$1 - p - q > 1 - \frac{n}{2} - \frac{n+1}{2} = -n + \frac{1}{2} \Rightarrow 1 - p - q \in [-n + 1; -1] \cap \mathbb{Z}$ Consequently there is a single h.

Analogously to • the maximum number of cells that can be coloured in blue in a column is either all cells with even y or all cells with odd y.

Problem 1

Let us examine a whole G(m, n) .

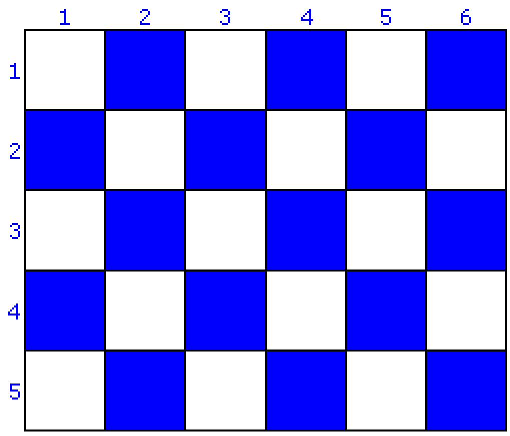


G(8,6)

ƒ If in the first column all the cells with odd y are coloured in blue, then in each row with an odd y we should colour all cells with an odd x, because the first cell of the row is coloured in blue, while in each row with an even y we should colour all the cells with even x, because the first cell is not coloured in blue.

Thus the number of the coloured in blue cells is:

1. $\frac{n}{2}.\frac{m}{2}+\frac{n}{2}.\frac{m}{2} = \frac{mn}{2}$ (where m and n are even)
2. $\frac{n+1}{2}.\frac{m}{2}+\frac{n-1}{2}.\frac{m}{2} = \frac{mn}{2}$ (where m is even and n is odd)
3. $\frac{n}{2}.\frac{m+1}{2}+\frac{n}{2}.\frac{m-1}{2} = \frac{mn}{2}$ (where m is odd and n is even)
4. $\frac{n+1}{2}.\frac{m+1}{2}+\frac{n-1}{2}.\frac{m-1}{2} = \frac{mn+m+n+1+mn-m-n+1}{4} = \frac{mn+1}{2}$ (where m and n are odd)



G(6,5)

„ If in the first column all the cells with even y are coloured in blue, then in each row with an odd y we should colour all cells with an even x, because the first cell of the row is not coloured in blue, while in each row with an even y we should colour all the cells with odd x, because the first cell is coloured in blue.

Thus the number of the coloured in blue cells is:

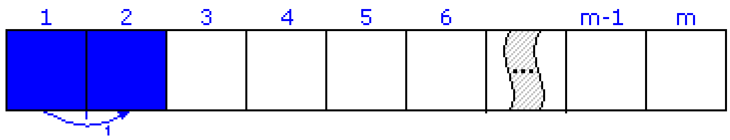
1. $\frac{n}{2}.\frac{m}{2}+\frac{n}{2}.\frac{m}{2} = \frac{mn}{2}$ (where m and n are even)
2. $\frac{n+1}{2}.\frac{m}{2}+\frac{n-1}{2}.\frac{m}{2} = \frac{mn}{2}$ (where m is even and n is odd)
3. $\frac{n}{2}.\frac{m+1}{2}+\frac{n}{2}.\frac{m-1}{2} = \frac{mn}{2}$ (where m is odd and n is even)
4. $\frac{n+1}{2}.\frac{m-1}{2}+\frac{n-1}{2}.\frac{m+1}{2} = \frac{mn+m-n-1+mn-m+n-1}{4} = \frac{mn-1}{2}$ (where m and n are odd)

From ƒ and „ we can conclude that the maximum number of cells that can be coloured in blue, while there is not a pair of symmetrical cells, is nm/2 , when nm is even and (nm+1)/2 , when nm is odd.

5 Problem 1

2. Let us examine the case when $n=1$, i.e. $G(m,n)$ consists of only one row.

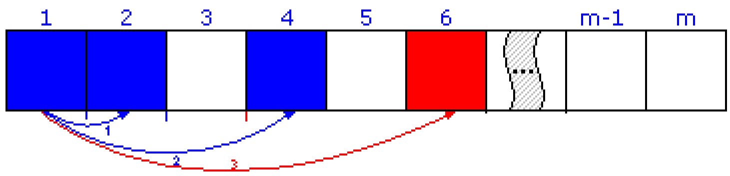
Let us start colouring cells in such way that in every colouring we use at least one of the unused vertical lines from left to right as an axis of symmetry. Thus on the first step we have to colour the first two cells.



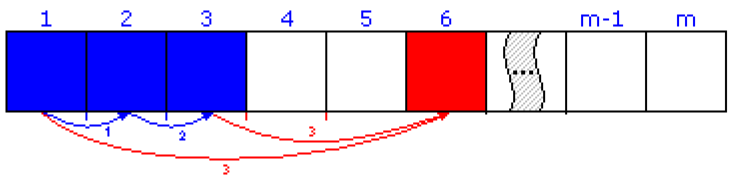
On the second step we can choose from 3 and 4, on the third, depending on the previous step – between 3, 5 and 6 or 4, 5 and 6 and etc.

Consequently there are two logical approaches:

• : We colour the farthest cell from the first in order to skip as many cells as possible. In this case in each step we are able to skip at most one cell, because as we colour a single cell, symmetrical to the first, we use only one vertical line as an axis of symmetry and we are compelled to use the next vertical line in the next step.



, : The second approach expands the idea of the first one, as we preliminarily colour a certain number of cells in the beginning of the row in order to be able to skip more (than one) cells in the next steps. It is necessary because with colouring the cell symmetrical to the first one in the row with respect to the first non-used vertical line we use as many vertical lines after it as there are cells with odd x in the beginning of the row.



In this solution we are going to examine the second approach.

Problem 1

Algorithm for colouring:

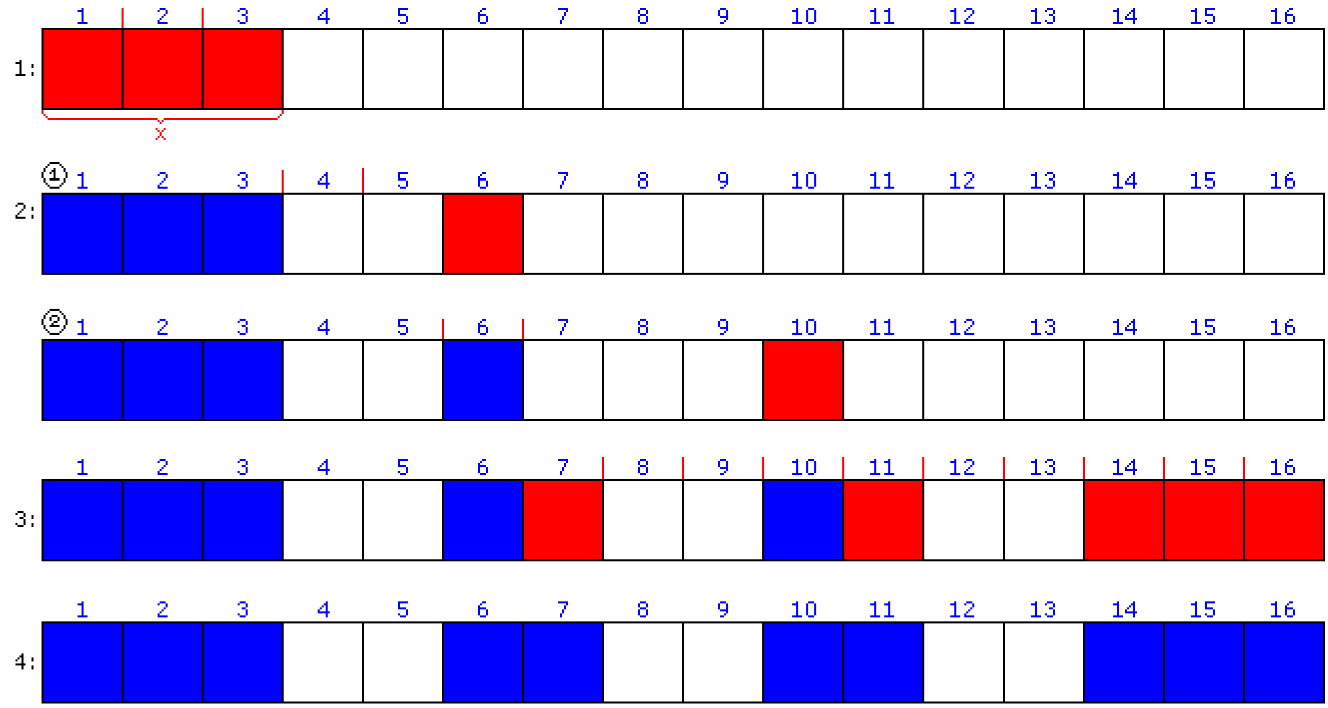
Step 1: We colour the first x cells of the row.

Substep a: We colour the first cell, symmetrical to the first one of the row with respect to the closest to the beginning of the row line that has not been used as an axis of symmetry.

Step 2: We execute Substep ak times.

Step 3: We colour all the cells symmetrical to the already coloured ones with respect to the middle of the row (Note: it is not needed to be a vertical line).

Step 4: k is such that the two central cells are not farther than the cells coloured in step 2.



The total number of coloured cells is $S(m,1) = 2(x+k)$ or $2(x+k)-1$.

The distance between two cells in step 2 is:

1: If x is odd: $x + 1$

2: If x is even: x

The first cell in step 2 is with x coordinate $2x$.

The last cell in step 2 is with x coordinate:

1: $2x+(k-1)(x+1) = x-1+kx+k = x+kx+k-1$ for x odd

2: $2x+(k-1)x = x+kx = x+kx$ for x even

Problem 1

The cell, symmetrical to the last cell coloured in step 2 with respect to the middle of the row, has x coordinate:

1: m+1–(x+kx+k–1) = m+1–x–kx–k+1 =m+2–x–kx–k for x odd

2: m+1–(x+kx) = m+1–x–kx = m+1–x–kx for x even

The difference between the last cell coloured in step 2 and its symmetrical with respect to the middle of the row is:

1: m+2–x–kx–k – (x+kx+k–1) =m+3–2(x+kx+k) for x odd

2: m+1–x–kx – (x+kx) = m+1–2(x+kx) for x even

Consequently from step 4:

1: m+3–2(x+kx+k) ≤ x+1
2: m+1–2(x+kx) ≤ x

1: m+2–3x–2kx–2k ≤ 0
2: m+1–3x–2kx ≤ 0

1: k.2(x+1) ≥ m+2–3x
2: k.2x ≥ m+1–3x

1: $k \geq \frac{m+2-3x}{2(x+1)}$ for an odd x

2: $k \geq \frac{m+1-3x}{2x}$ for an even x

Therefore k is the least possible integer that satisfies one of the previous inequalities.

*Note:

When 2(x + kx + k) = m + 3 for an odd x and 2(x + kx) = m + 1 for an even x then the distance between the last cell in step 2 and its symmetrical with respect to the middle of the row is 0 therefore they coincide and the total number of cells is 2 * (x + k) – 1.

When 2(x + kx + k) ≠ m + 3 for an odd x and 2(x + kx) ≠ m + 1 for an even x the total number of cells is 2 * (x + k).

Reasoning:

Steps 1 and 2 allow us to guarantee that we have used the first $\frac{x+kx+k-1}{2}$ vertical lines, when x is odd and $\frac{x+kx}{2}$ vertical lines, when x is even. After the third step, along with colouring the cells symmetrical to the ones that are already colored with respect to the middle, we use the vertical lines that are symmetrical to the ones that are already used with respect to the middle i.e. the last $\frac{x+kx+k-1}{2}$ vertical lines, when x is odd and $\frac{x+kx}{2}$ vertical lines, when x is even. The cells, symmetrical to the ones coloured in step 2 with respect to the middle of the row, have x coordinate m–2x, m–2x–(x+1), … , m – (x + kx + k – 1), when x is odd and m–2x, m–3x, … , m – (x + kx), when x is even. If m is odd these cells have odd x coordinate and if m is even they have even x coordinate. Thus they are respectively symmetrical to the first or the second cell of the grid which, as we know, are coloured in step 1. The vertical lines with respect to which they are symmetrical are:

1) for m even:

$v_i = \frac{1 + m - (x - 1 + i(x+1)) - 1}{2} = \frac{\boldsymbol{m - (x - 1 + i(x+1))}}{\mathbf{2}}$, when x is odd , i ≤ k.

$v_i = \frac{1 + m - (x + ix) - 1}{2} = \frac{\boldsymbol{m - (x + ix)}}{\mathbf{2}}$, when x is even, i ≤ k.

2) for m odd:

$v_i = \frac{2 + m - (x - 1 + i(x+1)) - 1}{2} = \frac{\boldsymbol{m + 1 - (x - 1 + i(x+1))}}{\mathbf{2}}$, when x is odd , i ≤ k.

$v_i = \frac{2 + m - (x + ix) - 1}{2} = \frac{\boldsymbol{m + 1 - (x + ix)}}{\mathbf{2}}$, when x is even, i ≤ k.

Thus the cell with the least x is $\frac{m - (x + kx + k - 1)}{2}$, $\frac{m - (x + ix)}{2}$, $\frac{m + 1 - (x + kx + k - 1)}{2}$ and $\frac{m + 1 - (x + kx)}{2}$, respectively. The difference between the x coordinates of this line and the last of the lines used for a symmetry in step 2 is:

$\frac{m - (x + kx + k - 1)}{2} - \frac{x + kx + k - 1}{2} = \frac{\boldsymbol{m}}{\mathbf{2}} - (\boldsymbol{x} + \boldsymbol{kx} + \boldsymbol{k} - \mathbf{1})$, for an odd x and even m;

$\frac{m - (x + kx)}{2} - \frac{x + kx}{2} = \frac{\boldsymbol{m}}{\mathbf{2}} - (\boldsymbol{x} + \boldsymbol{kx})$, for even m and x;

$\frac{m + 1 - (x + kx + k - 1)}{2} - \frac{x + kx + k - 1}{2} = \frac{\boldsymbol{m+1}}{\mathbf{2}} - (\boldsymbol{x} + \boldsymbol{kx} + \boldsymbol{k} - \mathbf{1})$, for odd m and x;

$\frac{m + 1 - (x + kx)}{2} - \frac{x + kx}{2} = \frac{\boldsymbol{m+1}}{\mathbf{2}} - (\boldsymbol{x} + \boldsymbol{kx})$, for an odd m and even x,

respectively.

We have already shown that:

m+3–2(x+kx+k) ≤ x+1 , for an odd x

m+1–2(x+kx) ≤ x , for an even x.

Therefore:

$\frac{\boldsymbol{m}}{\mathbf{2}} - (\boldsymbol{x} + \boldsymbol{kx} + \boldsymbol{k} - \mathbf{1}) = \frac{\boldsymbol{m + 2 - 2(x+kx+k)}}{\mathbf{2}} \leq \frac{\boldsymbol{x + 1 - 1}}{\mathbf{2}} = \frac{\boldsymbol{x}}{\mathbf{2}}$;

$\frac{\boldsymbol{m}}{\mathbf{2}} - (\boldsymbol{x} + \boldsymbol{kx}) = \frac{\boldsymbol{m - 2(x + kx)}}{\mathbf{2}} \leq \frac{\boldsymbol{x-1}}{\mathbf{2}}$;

$\frac{\boldsymbol{m+1}}{\mathbf{2}} - (\boldsymbol{x} + \boldsymbol{kx} + \boldsymbol{k} - \mathbf{1}) = \frac{\boldsymbol{m + 3 - 2(x+kx+k)}}{\mathbf{2}} \leq \frac{\boldsymbol{x+1}}{\mathbf{2}}$;

$\frac{\boldsymbol{m+1}}{\mathbf{2}} - (\boldsymbol{x} + \boldsymbol{kx}) = \frac{\boldsymbol{m+1 - 2(x+kx)}}{\mathbf{2}} \leq$ x.

Consequently the number of lines that we have not proved that are used as an axis of symmetry are placed symmetrically with respect to the middle of the row, are consecutive and their number is less than $\frac{x+1}{2}$. A cell with x coordinate is symmetric with respect to the lines:

m, m–2, ... , m–x+2 for even m and x;

m, m–2, ... , m–x+1 for even m and odd x;

m–1, m–3, ... , m–x+1 for odd m and even x;

m–1, m–3, ... , m–x+2 for odd m and x.

Namely, at least $\frac{x-1}{2}$ and not less than 1. They are placed before the line $\frac{m}{2}$.

Analogously a cell with number m is symmetric to at least $\frac{x-1}{2}$ cells from the first x cells in the beginning of the row and at least 1. They are placed after the line $\frac{m}{2}$. Thus these two groups are consecutive lines and form one common group, symmetrical with respect to the middle of the grid and with length more than 2 and at least $x-1 \geq \frac{x+1}{2}$ (when $x \geq 3$, and if $x = 2$, the lines are still at least $2 > \frac{x+1}{2}$).

Therefore all vertical lines are used for symmetry, i.e the algorithm produces specular colouring .

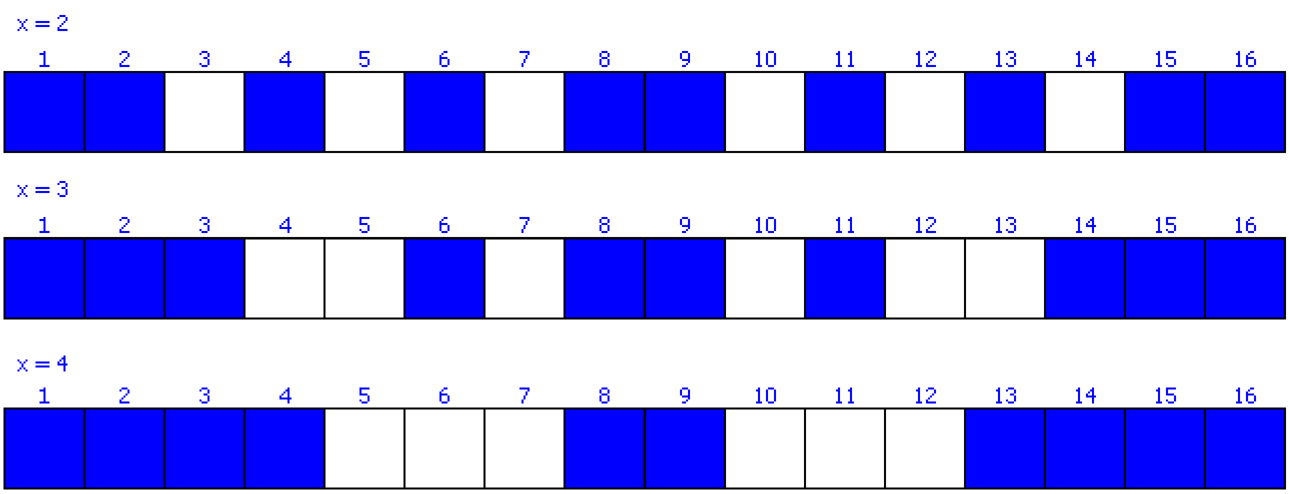
Example: For m = 16:

for x = 2: k ≥ (17 – 6)/(2.2) = 11/4 so k = 3
2(2 + 6 + 3) = 22 ≠ 19 so S(16, 1) is potentially 2 * (2 + 3) = 10.

for x = 3: k ≥ (18–9)/(2*4) = 9/8 so k = 2.
2(3 + 6 + 2) = 22 ≠ 19 so S(16, 1) is potentially 2 * (3 + 2) = 10.

for x = 4: k ≥ (17 – 12) / 8 = 5/8 so k = 1;
2(4 + 4 + 1) = 18 ≠ 19 so S(16,1) is potentially 2* (4 + 1) = 10

The minimal of the three numbers is 10 and therefore S(16, 1) = 10.



Problem 1

2(k+x) ≥

1: $2.\left(\frac{m+2-3x}{2.(x+1)}+x\right)=\frac{m+2-3x+2x^2+2x}{x+1}=\frac{2x^2-x+m+2}{x+1}=2x-3+\frac{m+5}{x+1}$ for x odd;

2: $2.\left(\frac{m+1-3x}{2x}+x\right)=\frac{m+1-3x+2x^2}{x}=\frac{2x^2-3x+m+1}{x}=2x-3+\frac{m+1}{x}$ for x even.

Therefore:

$$2(k+x)\geq f(x),\text{ where } f(x)=\begin{cases}2x-3+\frac{m+5}{x+1}, & where\ x\ is\ odd\\ 2x-3+\frac{m+1}{x}, & where\ x\ is\ even\end{cases}$$

Let $f_1(x)=2x-3+\frac{m+5}{x+1}$ for $x\in R$, x > 0

Therefore the derivative of $f_1(x)$, $f_1'(x)=\frac{2-m+5}{(x+1)^2}=\frac{2x^2+4x-m-3}{(x+1)^2}$.

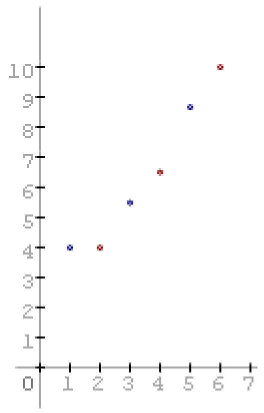
D' = 4+2m+6 = 10+2m

Therefore $x_{min1}=\frac{-2+\sqrt{10+2m}}{2}=-1+\frac{\sqrt{10+2m}}{2}$

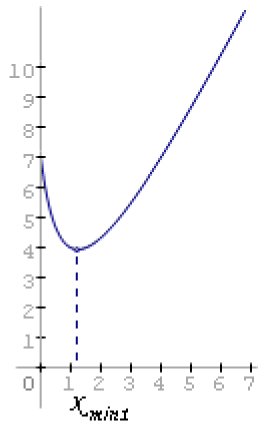
Let $f_2(x)=2x-3+\frac{m+1}{x}$ for $x\in R$, x > 0.

Therefore the derivative of $f_2(x)$, $f_2'(x)=2-\frac{m+1}{x^2}=\frac{2x2-m-1}{x^2}$ consequently $x_{min2}=\sqrt{\frac{m+1}{2}}$

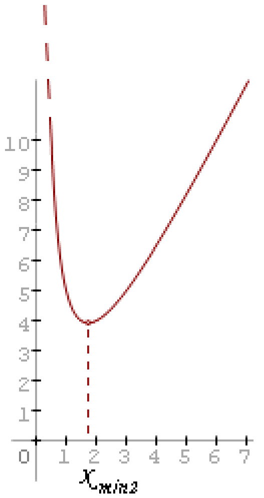
$f(x)$ for $m = 5$



$f_1(x)$ for $m = 5$



$f_2(x)$ for $m = 5$



Let x_1 be the greatest odd integer less than x_{min1} and x_2 be the least odd integer more than $\geq x_{min1}$.

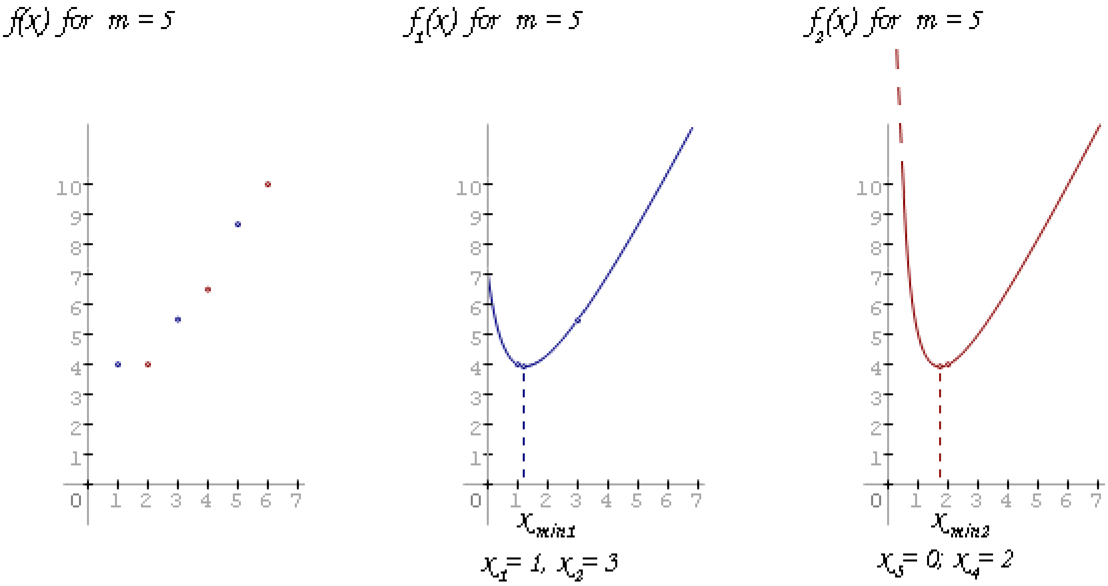
As $f_1(x)$ is monotonously decreasing for $x \in (0; x_{min1}.)$ and is monotonously increasing for $x \in (x_{min1}, +\infty)$, $f(x_1) = f_1(x_1) \geq f_1(x_{min1})$ and $f(x_2) = f_1(x_2) \geq f_1(x_{min1})$.

Let $f(x_1) = y_1$ and $f(x_2) = y_2$.

Let x_3 be the greatest even integer less than x_{min2} and x_4 be the least even integer more than x_{min2}.

As $f_2(x)$ is monotonously decreasing for $x \in (0; x_{min2}.)$ and is monotonously increasing for $x \in (x_{min2}, +\infty)$, $f(x_3) = f_2(x_3) \geq f_2(x_{min2})$ and $f(x_4) = f_2(x_4) \geq f_2(x_{min2})$.

Let $f(x_3) = y_3$ and $f(x_4) = y_4$.



As $2(x+k)$ is an integer, an odd number and $2(x+k) \geq f(x)$ then $2(x+k) = \min(\lceil y_1 \rceil, \lceil y_2 \rceil, \lceil y_3 \rceil, \lceil y_4 \rceil)$ (+1, when the result is an odd number).

($\lceil x \rceil$ is the smallest number greater than or equal to x).

In this way we can compute S(m, 1).

Example: S(104, 1)

$$x_{min1} = -1 + \frac{\sqrt{10+208}}{2} = -1 + \frac{\sqrt{218}}{2} \approx 6{,}38.$$

$$x_{min2} = \sqrt{\frac{1+104}{2}} = \sqrt{57{,}5} \approx 7{,}58.$$

$=> x_1 = 5; x_2 = 7; x_3 = 6; x_4 = 8.$

$$f(5) = 10 - 3 + \frac{104+5}{6} = 7 + 18 + \frac{1}{6} = 25\frac{1}{6}$$

$$f(6) = 12 - 3 + \frac{105}{6} = 9 + \frac{35}{2} = 26\frac{1}{2}$$

$$f(7) = 14 - 3 + \frac{109}{8} = 11 + 14 + \frac{5}{8} = 25\frac{5}{8}$$

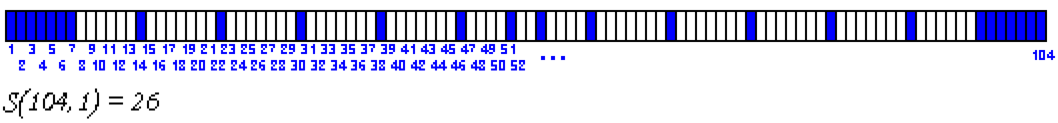
$$f(8) = 16 - 3 + \frac{105}{8} = 13 + 14 + \frac{1}{8} = 27\frac{1}{8}$$

Therefore $2(k+x) = 26$ when $x = 5$ or 7 and $k = 8$ or 6.

For $x = 5$ and $k = 8$: $2(5 + 40 + 8) = 104 \neq 104 + 3$;

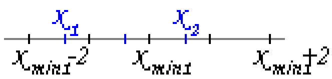
For $x = 7$ and $k = 6$: $2(7 + 42 + 6) = 106 \neq 104 + 3$;

So $S(104, 1) = 2(x + k) = 26$.



Problem 1

From the properties of x_1 and x_2 we can conclude that $x_1 > x_{min1} - 2$ and $x_2 < x_{min1} + 2$



So:

$x_1 > -3 + \frac{\sqrt{10+2m}}{2}$, $x_1 \le x_{min1}$. Thus:

$$y_1 < 2\left(-3 + \frac{\sqrt{10+2m}}{2}\right) + \frac{m+5}{-3+\frac{\sqrt{10+2m}}{2}+1}$$

$$y_1 < -6 + \sqrt{10 + 2m} + \frac{10+2m}{-4+\sqrt{10+2m}}$$

$$y_1 < -6 + \sqrt{10 + 2m} + \frac{(10+2m)(4+\sqrt{10+2m})}{10+2m-16}$$

and

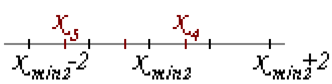
$x_2 < 1 + \frac{\sqrt{10+2m}}{2}$, $x_2 \ge x_{min1}$. Thus:

$$y_2 < 2\left(1 + \frac{\sqrt{10+2m}}{2}\right) + \frac{m+5}{1+\frac{\sqrt{10+2m}}{2}+1}$$

$$y_2 < 2 + \sqrt{10 + 2m} + \frac{10+2m}{4+\sqrt{10+2m}}$$

$$y_2 < 2 + \sqrt{10 + 2m} + \frac{(10+2m)(\sqrt{10+2m}-4)}{10+2m-16}$$

From the properties of x_3 and x_4 we can conclude that $x_3 > x_{min2} - 2$ and $x_4 < x_{min2} + 2$



So:

$x_3 > \sqrt{\frac{m+1}{2}} - 2$, $x_3 \le x_{min2}$. Thus:

$$y_3 < 2\left(\sqrt{\frac{m+1}{2}} - 2\right) - 3 + \frac{m+1}{\sqrt{\frac{m+1}{2}}-2}$$

$$y_3 < \sqrt{2(m+1)} - 7 + \frac{(2m+2)(\sqrt{2(m+1)}+4)}{(2m+2)-16}$$

and

$x_4 < x_{min2} + 2$, $x_4 \geq x_{min2}$. Thus:

$$y_4 < 2\left(\sqrt{\frac{m+1}{2}} + 2\right) - 3 + \frac{m+1}{\sqrt{\frac{m+1}{2}} + 2}$$

$$y_4 < \sqrt{2(m+1)} + 1 + \frac{(2m+2)\left(\sqrt{2(m+1)}-4\right)}{(2m+2)-16}$$

$$y_1, y_2 \geq y_{min1} = f_1(x_{min1}) = -2 + \sqrt{10+2m} - 3 + \frac{m+5}{\frac{\sqrt{10+2m}}{2}} = -5 + 2\sqrt{10+2m}$$

$$y_3, y_4 \geq y_{min2} = f_2(x_{min2}) = 2\sqrt{\frac{m+1}{2}} - 3 + \frac{m+1}{\sqrt{\frac{m+1}{2}}} = -3 + 2\sqrt{2m+2}$$

$$=> \quad \left|\begin{array}{l} y \geq y_1 \\ y < y_1 + 2 \end{array}\right. \quad \text{or} \quad \left|\begin{array}{l} y \geq y_2 \\ y < y_2 + 2 \end{array}\right. \quad \text{or} \quad \left|\begin{array}{l} y \geq y_2 \\ y < y_2 + 2 \end{array}\right. \quad \text{or} \quad \left|\begin{array}{l} y \geq y_2 \\ y < y_2 + 2 \end{array}\right.$$

$$S(m,1) \in \left[-6 + 2\sqrt{10+2m}; -4 + \sqrt{10+2m} + \frac{(10+2m)(4+\sqrt{10+2m})}{10+2m-16}\right) \cap$$

$$\cap\left[-6 + 2\sqrt{10+2m}; 4 + \sqrt{10+2m} + \frac{(10+2m)(\sqrt{10+2m}-4)}{10+2m-16}\right) \cap$$

$$\cap\left[-4 + 2\sqrt{2m+2}; -5 + \sqrt{2(m+1)} + \frac{(2m+2)(\sqrt{2(m+1)}+4)}{(2m+2)-16}\right) \cap$$

$$\cap\left[-4 + 2\sqrt{2m+2}; 3 + \sqrt{2(m+1)} + \frac{(2m+2)\left(\sqrt{2(m+1)}-4\right)}{(2m+2)-16}\right)$$

Thus $S(104,1) \in \left[-6 + 2\sqrt{218}; -4 + \sqrt{218} + \frac{218.(4+\sqrt{218})}{202}\right) \cap$

$$\cap\left[-6 + 2\sqrt{218}; 4 + \sqrt{218} + \frac{218.(\sqrt{218}-4)}{202}\right) \cap$$

$$\cap\left[-4 + 2\sqrt{210}; -5 + \sqrt{210} + \frac{210.(\sqrt{210}+4)}{194}\right) \cap$$

$$\cap\left[-4 + 2\sqrt{210}; 3 + \sqrt{210} + \frac{210.(\sqrt{210}-4)}{194}\right) =$$

$= \quad (23{,}52; 31{,}02) \cup (23{,}52; 30{,}39) \cup (24{,}98; 29{,}51) \cup (24{,}98; 28{,}85) =$

$= \quad (23{,}52; 31{,}02)$, which includes 26 => ok.

Following analogous reasoning we can limit, G(1,n) in the same way:

$$S(n,1) \in \left[-6 + 2\sqrt{10+2n};\ -4 + \sqrt{10+2n} + \frac{(10+2n)(4+\sqrt{10+2n})}{10+2n-16}\right) \cap$$
$$\cap\left[-6 + 2\sqrt{10+2n};\ 4 + \sqrt{10+2n} + \frac{(10+2n)(\sqrt{10+2n}-4)}{10+2n-16}\right) \cap$$
$$\cap\left[-4 + 2\sqrt{2n+2};\ -5 + \sqrt{2(n+1)} + \frac{(2n+2)(\sqrt{2(n+1)}+4)}{(2n+2)-16}\right) \cap$$
$$\cap\left[-4 + 2\sqrt{2n+2};\ 3 + \sqrt{2(n+1)} + \frac{(2n+2)\left(\sqrt{2(n+1)}-4\right)}{(2n+2)-16}\right)$$

If there is a grid G(m, n) and not all symmetries with respect to vertical lines are on a single row it is easy to see that if they are projected on one row the total number of cells does not increase so in the optimal colouring all the pairs of cells symmetrical with respect to a vertical line are on a single row.

Analogously all the pairs of cells symmetrical with respect to a horizontal line should be in a single column.

Therefore S(m, n) = S(m, 1) + S(1, n) – 1 (when a cell from the chosen column and row coincide).

$$S(m,n) = \left[-6 + 2\sqrt{10+2m};\ -4 + \sqrt{10+2m} + \frac{(10+2m)(4+\sqrt{10+2m})}{10+2m-16}\right) \cap$$
$$\cap\left[-6 + 2\sqrt{10+2m};\ 4 + \sqrt{10+2m} + \frac{(10+2m)(\sqrt{10+2m}-4)}{10+2m-16}\right) \cap$$
$$\cap\left[-4 + 2\sqrt{2m+2};\ -5 + \sqrt{2(m+1)} + \frac{(2m+2)(\sqrt{2(m+1)}+4)}{(2m+2)-16}\right) \cap$$
$$\cap\left[-4 + 2\sqrt{2m+2};\ 3 + \sqrt{2(m+1)} + \frac{(2m+2)\left(\sqrt{2(m+1)}-4\right)}{(2m+2)-16}\right)$$
$$+$$
$$\left[-6 + 2\sqrt{10+2n};\ -4 + \sqrt{10+2n} + \frac{(10+2n)(4+\sqrt{10+2n})}{10+2n-16}\right) \cap$$
$$\cap\left[-6 + 2\sqrt{10+2n};\ 4 + \sqrt{10+2n} + \frac{(10+2n)(\sqrt{10+2n}-4)}{10+2n-16}\right) \cap$$
$$\cap\left[-4 + 2\sqrt{2n+2};\ -5 + \sqrt{2(n+1)} + \frac{(2n+2)(\sqrt{2(n+1)}+4)}{(2n+2)-16}\right) \cap$$
$$\cap\left[-4 + 2\sqrt{2n+2};\ 3 + \sqrt{2(n+1)} + \frac{(2n+2)\left(\sqrt{2(n+1)}-4\right)}{(2n+2)-16}\right) - 1.$$

3. Formulate and investigate 3-dimensional analogs of the problem.

1. What is the maximum number of cubes of an $m \times n \times t$ grid that can be coloured blue, such that no two blue cubes are symmetric with respect to any (mn), (mt) or (nt) planes of the grid?

2. Some cubes of an $m \times n \times t$ grid are coloured blue. We call such a colouring specular if for any interior (mn), (mt) or (nt) plane of the grid there are two blue cubes that are symmetric with respect to this plane. Denote by $S(m, n, t)$ the minimal number of blue cubes in a specular colouring of an $m \times n \times t$ grid.
Find $S(m, n, t)$, or estimate it (give lower and upper bounds).

In the first part of the current problem the solution is analogous to the 2-dimensional version.

If m.n.t is an even number then the maximum number of cubes is $\frac{m.n.t}{2}$
If m.n.t is an odd number then the maximum number of cubes is $\frac{m.n.t+1}{2}$

In the second part of the current problem, for every dimension the given inequalities in the 2-dimensional version are fulfilled where in the place of m or n lies the representative of the current dimension:

Thus in the 3-dimensional version with t, representing the third dimension, we have:

$$S(m,n,t) = \Big[-6 + 2\sqrt{10 + 2m}; -4 + \sqrt{10 + 2m} + \frac{(10+2m)(4+\sqrt{10+2m})}{10+2m-16}\Big) \cap$$
$$\cap\Big[-6 + 2\sqrt{10 + 2m}; 4 + \sqrt{10 + 2m} + \frac{(10+2m)(\sqrt{10+2m}-4)}{10+2m-16}\Big) \cap$$
$$\cap\Big[-4 + 2\sqrt{2m+2}; -5 + \sqrt{2(m+1)} + \frac{(2m+2)(\sqrt{2(m+1)}+4)}{(2m+2)-16}\Big) \cap$$
$$\cap\Big[-4 + 2\sqrt{2m+2}; 3 + \sqrt{2(m+1)} + \frac{(2m+2)\left(\sqrt{2(m+1)}-4\right)}{(2m+2)-16}\Big)$$
$$+$$
$$\Big[-6 + 2\sqrt{10 + 2m}; -4 + \sqrt{10 + 2m} + \frac{(10+2m)(4+\sqrt{10+2m})}{10+2m-16}\Big) \cap$$
$$\cap\Big[-6 + 2\sqrt{10 + 2m}; 4 + \sqrt{10 + 2m} + \frac{(10+2m)(\sqrt{10+2m}-4)}{10+2m-16}\Big) \cap$$
$$\cap\Big[-4 + 2\sqrt{2m+2}; -5 + \sqrt{2(m+1)} + \frac{(2m+2)(\sqrt{2(m+1)}+4)}{(2m+2)-16}\Big) \cap$$
$$\cap\Big[-4 + 2\sqrt{2m+2}; 3 + \sqrt{2(m+1)} + \frac{(2m+2)\left(\sqrt{2(m+1)}-4\right)}{(2m+2)-16}\Big)$$
$$+$$
$$\Big[-6 + 2\sqrt{10 + 2n}; -4 + \sqrt{10 + 2n} + \frac{(10+2n)(4+\sqrt{10+2n})}{10+2n-16}\Big) \cap$$
$$\cap\Big[-6 + 2\sqrt{10 + 2n}; 4 + \sqrt{10 + 2n} + \frac{(10+2n)(\sqrt{10+2n}-4)}{10+2n-16}\Big) \cap$$
$$\cap\Big[-4 + 2\sqrt{2n+2}; -5 + \sqrt{2(n+1)} + \frac{(2n+2)(\sqrt{2(n+1)}+4)}{(2n+2)-16}\Big) \cap$$
$$\cap\Big[-4 + 2\sqrt{2n+2}; 3 + \sqrt{2(n+1)} + \frac{(2n+2)\left(\sqrt{2(n+1)}-4\right)}{(2n+2)-16}\Big) \quad - \quad 2.$$

Problem 1

According to the results we can conclude that an n-dimensional analogue can be defined. If the representatives of the dimensions are d_1, d_2, ... , d_n, respectively, then:

If $d_1.d_2. \ldots .d_n$ is an even number then the maximum number of coloured cubes is $\frac{d_1.d_2 \ldots d_n}{2}$.

If $d_1.d_2. \ldots .d_n$ is an odd number then the maximum number of coloured cubes is $\frac{d_1.d_2 \ldots d_n + 1}{2}$.

In the second part of the current problem

$$S(d_1, d_2, \ldots, d_n) = \sum_{i=1}^{n} \left(\begin{array}{c} \left[-6 + 2\sqrt{10 + 2d_i}; -4 + \sqrt{10 + 2d_i} + \frac{(10 + 2d_i)\left(4 + \sqrt{10 + 2d_i}\right)}{10 + 2d_i - 16}\right) \cap \\ \left[-6 + 2\sqrt{10 + 2d_i}; 4 + \sqrt{10 + 2d_i} + \frac{(10 + 2d_i)\left(\sqrt{10 + 2d_i} - 4\right)}{10 + 2d_i - 16}\right) \cap \\ \left[-4 + 2\sqrt{2d_i + 2}; -5 + \sqrt{2(d_i + 1)} + \frac{(2d_i + 2)\left(\sqrt{2(d_i + 1)} + 4\right)}{(2d_i + 2) - 16}\right) \cap \\ \left[-4 + 2\sqrt{2d_i + 2}; 3 + \sqrt{2(d_i + 1)} + \frac{(2d_i + 2)\left(\sqrt{2(d_i + 1)} - 4\right)}{(2d_i + 2) - 16}\right) \end{array} \right) - n + 1.$$