Minsk, Belorus

Problem 7
Placements of Pentominoes

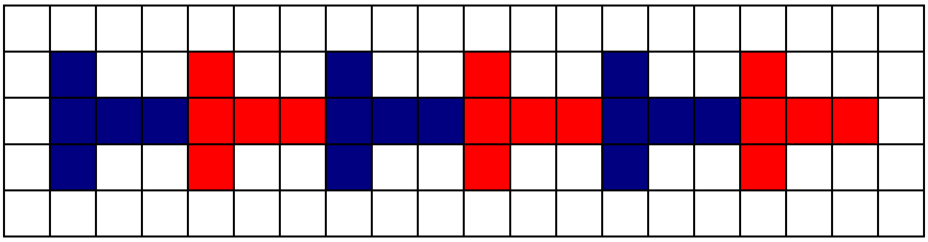
Contents:

1. The statement of the problem.

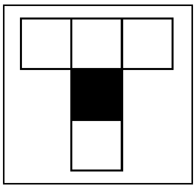
Given an *m* x *n* rectangle, denote by *T*(*m*, *n*) the minimum number of non-overlapping pentominoes that must be placed (along the grid lines) so that there is no place on the free cells for another pentomino. Find or estimate the number*T*(*m*, *n*) and give an algorithm for constructing suitable placements.

2. Rectangle 5 x *n*.

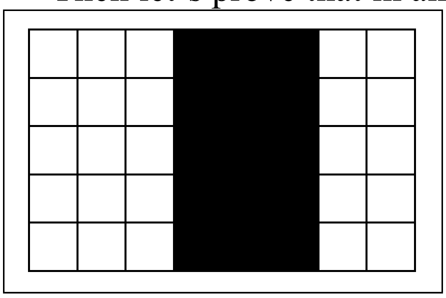
First let's solve the problem for rectangle 5 x*n*:



Denote the following cell of pentomino as its*centre*:



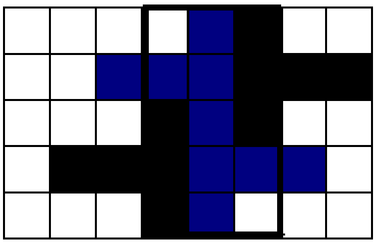
Then let's prove that in any rectangle 5 x 3 placed like this



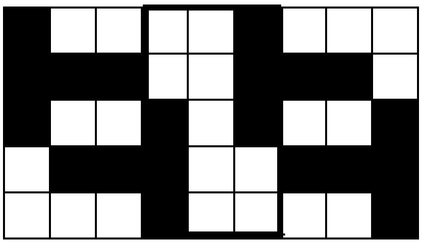
there must be at least one centre of pentomino. Suppose there exists a rectangle 5 x 3 that it doesn't contain any centres. Then:

A		A
B		B
C		C
D		D
E		E

One of any two cells denoted by the same letter must be “blocked”. There's the only way to do that (denoted black):



We also have to block two dispositions of pentomino denoted blue:



Then four pentominoes block rectangle 5 x 9 (and with optimal disposition – 5 x 12).

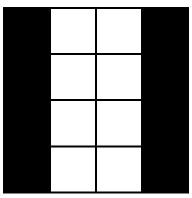
We use $[\frac{n}{3}]$ pentominoes.

3. Rectangle 4x*n*.

If the centre of one pentomino lies in the *k*-th column and centre of another pentomino lies in the *l*-th column, then call $|k-l|$ the *distance* between this pentaminoes.

Let’s enumerate all pentominoes such way that if the centre of one pentomino is in the *k*-th column and centre of another pentomino is in the *l*-th column and $k < l$, then the index of first pentomino is less than the index of the second.

Prove that there exists no pair of pentominoes (№№ *n* and *n* + 1) such that the distance between them exceeds 4. Suppose there exist a pair of pentominoes like this:

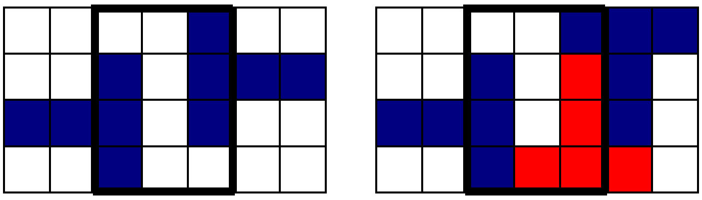


Then there exists a rectangle 4 x 4 without centres of pentominoes (shown on the picture). Then, any black cell must be “blocked”, and one can trivially obtain that it’s impossible.

Suppose the distance between the *n*-th and *n* + 1-st pentominoes equals 4. Then prove that distances between *n*-1-st and *n*-th pentominoes and between *n* + 1-st and $n + 2$-nd pentominoes are less then 4. There is a rectangle 4 x 3 that doesn't contain any centres in it:

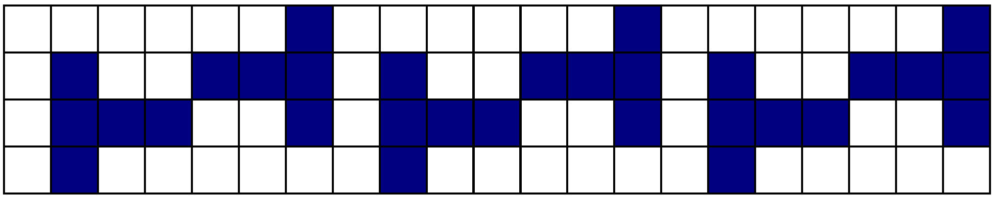
A		A
B		B
C		C
D		D

One of any two cells denoted by one letter must be "blocked". There are two ways to do that (denoted blue):



In the first case it's easy to see that distances between n - 1-st and n-th pentominoes and between n+ 1 and $n + 2$ pentominoes are less than 4. In the second case we can place a pentamino between n-th and $n + 1$-st pentaminoes.

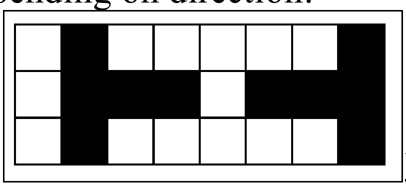
Then obviously the following disposition is optimal:



If $n = 7k$ or $n = 7k + 1$, then we use $2k$ pentominoes; if $n = 7k + 2$, $n = 7k + 3$ or $n = 7k + 4$ then we use $2k + 1$ pentominoes; if $n = 7k + 5$ or $n = 7k + 6$ then we use $2k + 2$ pentominoes. (EXEPTION: if n=2 we use 0 pentominoes, if n=5 we use 1 pentomino)

4) Rectangle 3 x *n*:

Let's find the minimum distance between$n$-th and n+1-st pentominoes depending on direction: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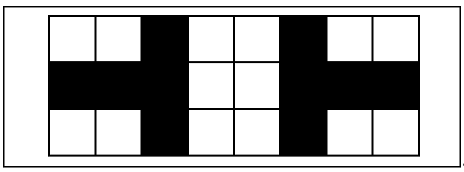
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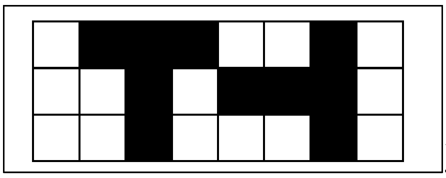
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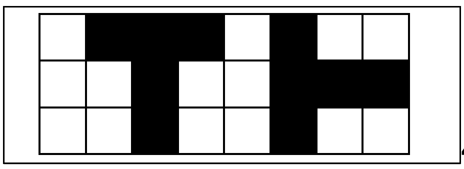
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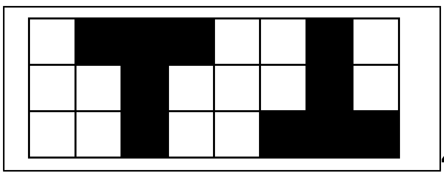
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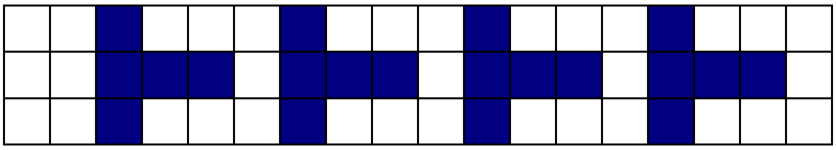
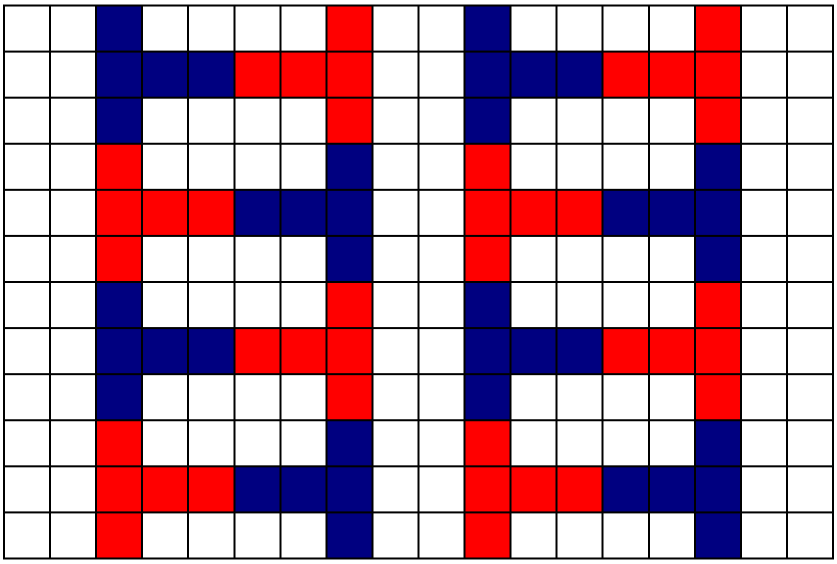
Then it's easy to prove that if there are two pairs of pentominoes (№№ k, k+1, l, l+1, $k < l$) such that the distance between k-th and k+1-st pentominoes and distance between l-th and l+1-st pentominoes is 5 then there exists the number m ($k < m < l$) such that the distance between m-th and m+1-st pentominoes is equal to 3.Then the following disposition is optimal:



We use $[\frac{4k+2}{4}]$ pentominoes.

5) Rectangle *m* x *n*.

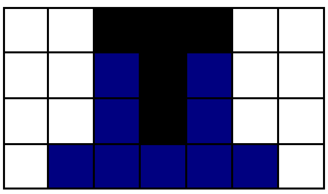
Algorithm for arbitrary m and n:



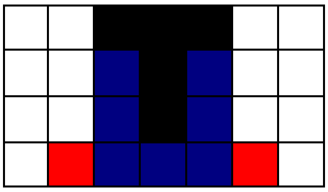
In this case we use a ´ b pentominoes, where:

1) $a = 2k$, if $m = 8k - 2$, $m = 8k - 1$ $m = 8k$, $m = 8k + 1$ or $m = 8k + 2$ and $a = 2k + 1$ in other cases;
2) $b = k$ if $n = 3k$, b = k + 1 if $n = 3k + 1$ and $b = k + 2$ if $n = 3k + 2$.

It's easy to see that there exist 4(m - 2)(n - 2) dispositions of pentomino in rectangle m x n (there is (m - 2)(n - 2) rectangles 3 x 3 and 4 dispositions of pentomino in each of them). Any pentomino "blocks" 64 dispositions of pentominoes.



Then, for any pentomino consider two more pentominoes (coloured blue on the picture). Each of them must be blocked. If we block both of them with one pentomino (if this pentomino will block the cell which belongs to both of two blue pentominoes), then both of this pentomino and the initialpentomino block not less than 4 dispositions of pentominoes.



Then, if we block two blue pentominoes with two differentpentominoes, then each of them “block” not less than 2 dispositions of pentominoes (in case they block two red cells), which were already blocked by the initial pentomino. Therefore not less than 4 of 64 dispositions which were blocked by one pentomino, will be blocked by another one. Then every pentomino blocked not more than 62 dispositions of pentominoes and we have to use $\frac{(m-2)(n-2)}{15,5}$ pentominoes.