Minsk, Belarus 2009

PATTERN GRAPHS

0. Basic definitions.

Let n be a positive integer. A *pattern of length n* is a two-line table $\begin{matrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{matrix}$, where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are some rearrangements of the numbers 1, 2, …, n.

Define two operations on patterns as follows

A: replace each number a of the first line with the number that is in the a'th place (from left to right) of the second line,

B: replace each number b of the second line with the number that is in the b'th place (from left to right) of the first line.

We can construct an oriented labelled graph G_n whose vertices are all the patterns of length n, and such that for any two vertices v and w there is an A-arrow (resp. a B-arrow) from v to w if the pattern w is obtained from the pattern v by applying the operation A (resp. the operation B).

Call the pattern $\begin{matrix} 1 & 2 & \ldots & n \\ 1 & 2 & \ldots & n \end{matrix}$ *unit*. Denote it as *ID*.

Orbit of a pattern X is the connected component of G_n, which this pattern belongs to.

Represent every pattern $X = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}$ as a pair of permutations (s, t), where $s = \begin{pmatrix} 1 & 2 & \ldots & n \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}$, $t = \begin{pmatrix} 1 & 2 & \ldots & n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}$. We can set a bijection between the set of patterns and the set of pairs of permutations of equal length. Denote this bijection with equality sign. It's easy to see, that operation A convert a pair (s, t) to the pair (ts, t), and operation B converts a pair (s, t) to the pair (s, st). Thus, if we define $A^k(X) = \underbrace{AA\ldots A}_{k}(X)$,

$A^{-k}(X) = Y \big| A^k(Y) = X$, and $A^0(X) = X$ (we define the powers of operation B in the same way), then $\forall k \in Z \quad A^k(s,t) = (t^k s, t), B^k(s,t) = (s, s^k t)$.

Call the pattern X *homogeneous*, if $X = (s^a, s^b)$, where s is an arbitrary permutation of degree n, $a, b \in Z_n$. It's obvious to see that the whole orbit of a homogeneous pattern (call such an orbit *homogeneous*) consists of homogeneous patterns.

Denote the set of prime numbers as P.

1. Connected components.

For every pattern $X = (s, t)$ consider the permutation group $G_X = \langle s, t \rangle$.

Lemma 1. The group G_X is invariant under the action of the operations A and B.

Proof. Really, $G_{A(X)} = \langle ts, t \rangle = \langle t^{-1}ts, t \rangle = \langle s, t \rangle$; $G_{B(X)} = \langle s, st \rangle = \langle s, s^{-1}st \rangle = \langle s, t \rangle$.

Lemma is proved.

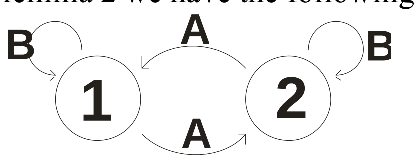
Theorem 1. If s,t are even permutations and at least one of the permutations x,y is odd? then the patterns $X=(s,t)$ and $Y=(x,y)$ lie in different connected components of G_n.
Proof. One of the permutations x,y is odd and belongs to G_Y. On the other hand, G_X is a subgroup of A_n, i.e. it doesn't contain odd permutations. Therefore $G_X \neq G_Y$, and the patterns X and Y lie in different connected components of G_n. Q.E.D.

Corollary 1. For every odd n the patterns $\begin{matrix} 2 & 1 & 3 & 4 & \ldots & n-1 & n \\ 2 & 3 & 4 & 5 & \ldots & n & 1 \end{matrix}$ and $\begin{matrix} 2 & 3 & 1 & 4 & \ldots & n-1 & n \\ 2 & 3 & 4 & 5 & \ldots & n & 1 \end{matrix}$ lie in different connected components of G_n, i.e. one cannot obtain one from another.

Lemma 2. Consider a pattern $X=(s,t)$. If $|s|=a, |t|=b$, then for any integer m and n such that $(m/b),(n/a) \notin Z$ $A^m(X) \neq B^n(X)$.
Proof. Notice that $A^m(s,t)=(t^m s,t)$, $B^n(s,t)=(s,s^n t)$.
Suppose that for some m and n the equality holds. Then $(t^m s,t)=(s,s^n t)$, that is equivalent to the system of equalities $\begin{cases} t^m s = s \\ s^n t = t \end{cases}$. Hence $s=t=id$, where id is the unity permutation, i.e. $X=$ ID – contradiction. Our assumption was wrong. Q.E.D.

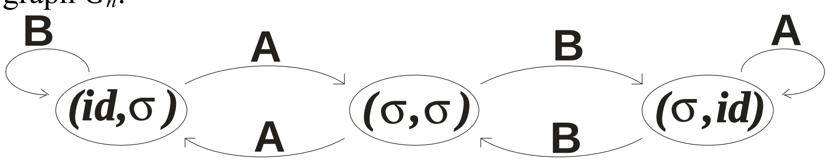
In other words, lemma 2 states that an arbitrary A-cycle and B-cycle have not more than one common pattern.

Theorem 2. G_n doesn't contain connected components of strength 2.
Proof. Suppose that G_n contains a connected component of strength 2. Then this component has either A-cycle or B-cycle of length 2. Without loss of generality, we have A-cycle. Then by lemma 2 we have the following picture:

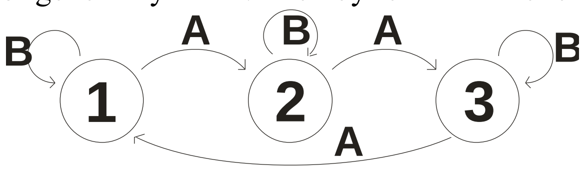


If the pattern 1 is equal to a pair of permutations (s,t), then $s=t^2=id$, and the pattern 2 is equal to the pair of permutations (t,t). But $B(2)=2$, whence $t=id$. Therefore the pattern is unit, but the unit pattern is situated in the component consisting of the only pattern – contradiction. Thus G_n doesn't contain connected components of strength 2. Q.E.D.

Theorem 3. G_n contains exactly $\sum_{k=1}^{[n/2]} \frac{n!}{2^k \cdot (n-2k)! \cdot k!}$ connected components of strength 3.
Proof. Notice that for every involution s there is a connected component of strength 3 in the graph G_n:



Show that there are no other connected components of strength 3 in G_n. Assume the contrary. If the component doesn't contain an X-cycle (where $X \in \{A,B\}$) then trivially obtain the component shown above. Therefore the component contains an X-cycle of length 3. Without loss of generality $X = A$. Then by lemma 2 the lengths of the other cycles are equal to 1:



Then pattern 1 is equal to (id, t), pattern 2 is equal to (t, t), and pattern 3 equals (t^2, t). But since $B(2) = 2$, then $t^2 = id$. Consequently, patterns 1 and 3 are equal – contradiction. Therefore there are no connected component differing from the one shown above in G_n.
The amount of required connected components in G_n is equal to the amount of involutions in S_n, which is equal to $\sum_{k=1}^{[n/2]} \frac{n!}{2^k \cdot (n-2k)! \cdot k!}$ (well-known fact). Q.E.D.

Since the study of patterns in general case is rather complicated, let's analyse some particular cases.

2. Homogeneous patterns.

Consider homogeneous patterns, which look like $X = (s^a, s^b)$, where s is an arbitrary permutation of degree n, $a, b \in Z_n$. If $(n, a, b) = k$, then we can reduce these numbers by k, considering the permutation s^k of degree n / k. From now on we consider the orbits of such homogeneous patterns, that $(n, a, b) = 1$. One can trivially prove that $A(s^a, s^b) = A(s^{a+b}, s^b)$, $B(s^a, s^b) = B(s^a, s^{a+b})$. From now on we calculate permutation's degrees modulo n, if nothing else is specified.
Theorem 4. The pattern (s^k, s^l) lies in the orbit of the pattern (s^a, s^b) if and only if $(n, k, l) = 1$, i.e. it's homogeneous.
Proof. Suppose we can obtain the pattern (s^a, s^b) from the pattern (s^k, s^l) using the operations A and B. If $(n, k, l) = p$, then $(n, a, b) \vdots p$, but $(n, a, b) = 1$. Therefore $(n, k, l) = 1$. Necessity is proved.
Suppose that $(n, k, l) = 1$. Let's prove a preliminary proposition.
Proposition 1. $\exists x \in Z | c = k + lx; (c, n) = 1$.
Proof. Divide n by all its common prime divisors with l. While dividing, take every prime in the maximal power it enters into n. Then we obtain some integer $n'; (n', l) = 1$. Consider the equality $c = k + lx$ modulo n'. Enumerating all the values of x from 0 to $(n' - 1)$, one can obtain all the possible values of c (since $(n', l) = 1$), therefore we can obtain some c_1 such that $(c_1, n') = 1$. Now consider the equality $c_1 = k + lx_1$ modulo n. l is the multiple of all the prime divisors of n/n' by construction. Since $(n, k, l) = 1$, then $(k, n/n') = 1$. We obtain that for every prime $p | n/n'$ $lx_1 \vdots p, (k, p) = 1$, therefore $(c_1, p) = 1$. Since it's true for every prime divisor of n/n', then $(c_1, n/n') = 1$. Since $(c_1, n') = 1$, then consequently $(c_1, n) = 1$. Thus we've shown that there exists such x_1, that $c_1 = k + lx_1; (c_1, n) = 1$.■
Thus, $\exists x \in Z | c = k + lx; (c, n) = 1$, and $A^x(s^k, s^l) = (s^c, s^l)$. Notice that $\exists p \in Z | z = b - pa; (z, n) = 1$ as well (the proof is similar to the proof of proposition 1). Then,

since $(c,n)=1$, we can obtain the pair (s^{c}, s^{z}) from the pair (s^{c}, s^{l}) using some operations B. As $(z,n)=1$, we can obtain the pair (s^{a}, s^{z}) from the pair (s^{c}, s^{z}) using some operations A. But $B^{p}(s^{a}, s^{z}) = (s^{a}, s^{b})$. Therefore the pattern (s^{a}, s^{b}) lies in the orbit of the pattern (s^{k}, s^{l}). Sufficiency is proved.
The theorem is proved.

In other words, theorem 4 claims that all the homogeneous patterns generated by the same permutation lie in one connected component.
We can calculate the quantity of patterns in the orbit of an arbitrary homogeneous pattern (s^{a}, s^{b}) using theorem 4. Denote this quantity by $q(s)$. Summing up the number of pairs for every possible GCD of n and a, obtain:

$$q(s) = \sum_{a_i|n}\left(j(n/a_i)\cdot j\left(\prod_{b_i|a_i, b\in P} b_i^{p_i}\right)\cdot \frac{n}{\prod_{b_i|a_i, b\in P} b_i^{p_i}}\right) = n\cdot\sum_{a_i|n}\left(j(n/a_i)\cdot \frac{j\left(\prod_{b_i|a_i, b\in P} b_i^{p_i}\right)}{\prod_{b_i|a_i, b\in P} b_i^{p_i}}\right),$$

where p_i are the maximal powers of β_i entering n. For example, for $n = p^{k}, p\in P$:

$$q(s) = n\cdot\sum_{i=1}^{k}\left(j(p^{k-i})\cdot\frac{j(n)}{n}\right) + n\cdot j(p^{k}) = j(p^{k})\cdot\sum_{i=1}^{k} j(p^{k-i}) + (p^{2k} - p^{2k-1}) =$$

$$= (p^{k} - p^{k-1})\cdot\left(\sum_{i=1}^{k-1}(p^{k-i} - p^{k-i-1}) + j(1)\right) + (p^{2k} - p^{2k-1}) = (p^{2k-1} - p^{2k-2}) + (p^{2k} - p^{2k-1}) =$$

$$= p^{2k} - p^{2k-2}.$$

We can also calculate the quantity of connected components of G_n consisting of homogeneous patterns of degree p.
Lemma 3. The orbits of two permutations s and t of degree p are equivalent if and only if $\exists a | (a,p)=1, t^{a} = s$.
Proof. Suppose that the orbits of two different permutations s and t of degree p are equivalent. Then $\exists a,b | (a,p)=(b,p)=1, s^{a} = t^{b}$, and $\exists x | ax=1 \Rightarrow (x,p)=1, s = t^{bx}$. We obtained that the permutation t has the required form, because $(bx,p)=1$.
Now we will prove that all the orbits of homogeneous patterns generated by permutations $s^{x}, (p,x)=1$, are equivalent. Really, $\left((s^{x})^{a}, (s^{x})^{b}\right) = (s^{xa}, s^{xb}); (p, xa, xb)=1$. Since x and p are coprimes, then we can always find some a_1 and b_1 such that $(s^{a}, s^{b}) = (s^{xa_1}, s^{xb_1}) = \left((s^{x})^{a_1}, (s^{x})^{b_1}\right)$. The correspondence is specified, therefore the orbits are equivalent.
The lemma is proved.
Using this lemma, we obtain, that the quantity of connected components of G_n consisting of homogeneous patterns of degree p equals $\frac{N(n,p)}{j(p)}$, where $N(n,p)$ - the quantity of permutations of degree p in the group S_n. In the case when p is prime, we can calculate $N(n,p)$ (the same way the quantity of involutions is calculated) and the required quantity:

$$\frac{N(n,p)}{j(p)} = \frac{\sum_{k=1}^{[n/p]}\frac{n!}{p^{k}\cdot(n-pk)!\cdot k!}}{p-1} = \sum_{k=1}^{[n/p]}\frac{n!}{p^{k}\cdot(n-pk)!\cdot k!\cdot(p-1)}.$$

3. The properties of G_n.

It's obvious to see that G_n is Euler graph (by the criterion of Euler graph: for every vertex the amount of entering arrows equals the amount of exiting arrows).

Theorem 5. Non-oriented graph G_n is invariant with respect to reassigning the A-edges and B-edges.

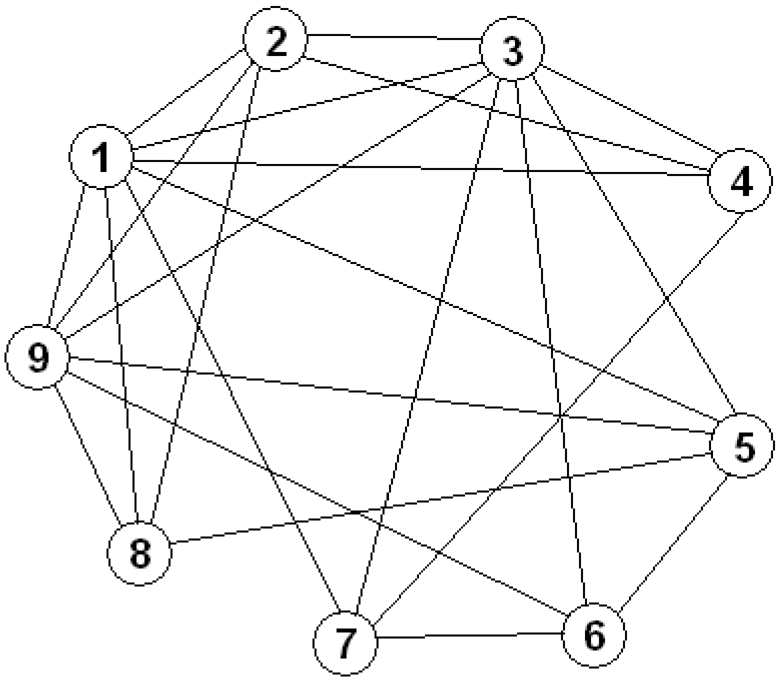
Proof. Introduce the map of vertices of G_n $C(X) = A^{-2}BA^{-1}B^{-1}(X)$. It's easy to show that this map is bijection. Then for $X = (s,t)$ we have $C(s,t) = (t^{-1}, t^{-1}st)$. Act on the vertices of the graph with the described map and trace where the images of its edges will be.

$$C(s,t) = (t^{-1}, t^{-1}st)$$
$$C(A(s,t)) = C(ts,t) = (t^{-1}, st) = B^{-1}(C(s,t))$$
$$C(B(s,t)) = C(s,st) = (t^{-1}s^{-1}, t^{-1}st) = A^{-1}(C(s,t))$$

Thus we have that the image of every A-arrow is reversed B-arrow and vice versa. Not taking into consideration the direction of the edges, we obtain the graph with the similar structure, but with A- and B-edges reassigned, therefore G_n is invariant with respect to reassigning the A-edges and B-edges. Q.E.D.

Theorem 6. The graph G_n is planar if and only if $n \le 3$.

Proof. It's obvious to show that for all $n \le 3$ G_n is planar. Consider the orbit of the pattern $\begin{matrix} 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{matrix}$ (see Fig. 1). Reduce all the A-arrows so that every A-cycles becomes one vertex. Then we obtain the following graph:



where vertex 1 is the A-cycle which second line was 2 3 4 1, vertex 2 – with the second line 1 3 4 2, vertex 3 – with the line 2 4 1 3, vertex 4 – with the line 2 1 3 4, vertex 5 – with the line 3 1 2 4, vertex 6 – with the line 1 4 3 2, vertex 7 – with the line 4 2 1 3, vertex 8 – with the line 4 2 3 1, vertex 9 – with the line 3 4 2 1. But this graph contains a section graph consisting of vertices 1, 2, 3, 5, 7, 9, which is isomorphic to the graph $K_{3,3}$. Then by Kuratowski's theorem graph G_4 is not planar. Notice that for every $n > 4$

there exists a pattern $\begin{matrix} 2 & 1 & 3 & 4 & 5 & \ldots & n \\ 2 & 3 & 4 & 1 & 5 & \ldots & n \end{matrix}$, which orbit is obviously isomorphic to the orbit of the pattern considered above. Therefore, for every $n \geq 4$ G_n is not planar. Thus, G_n is planar if and only if $n \leq 3$. Q.E.D.

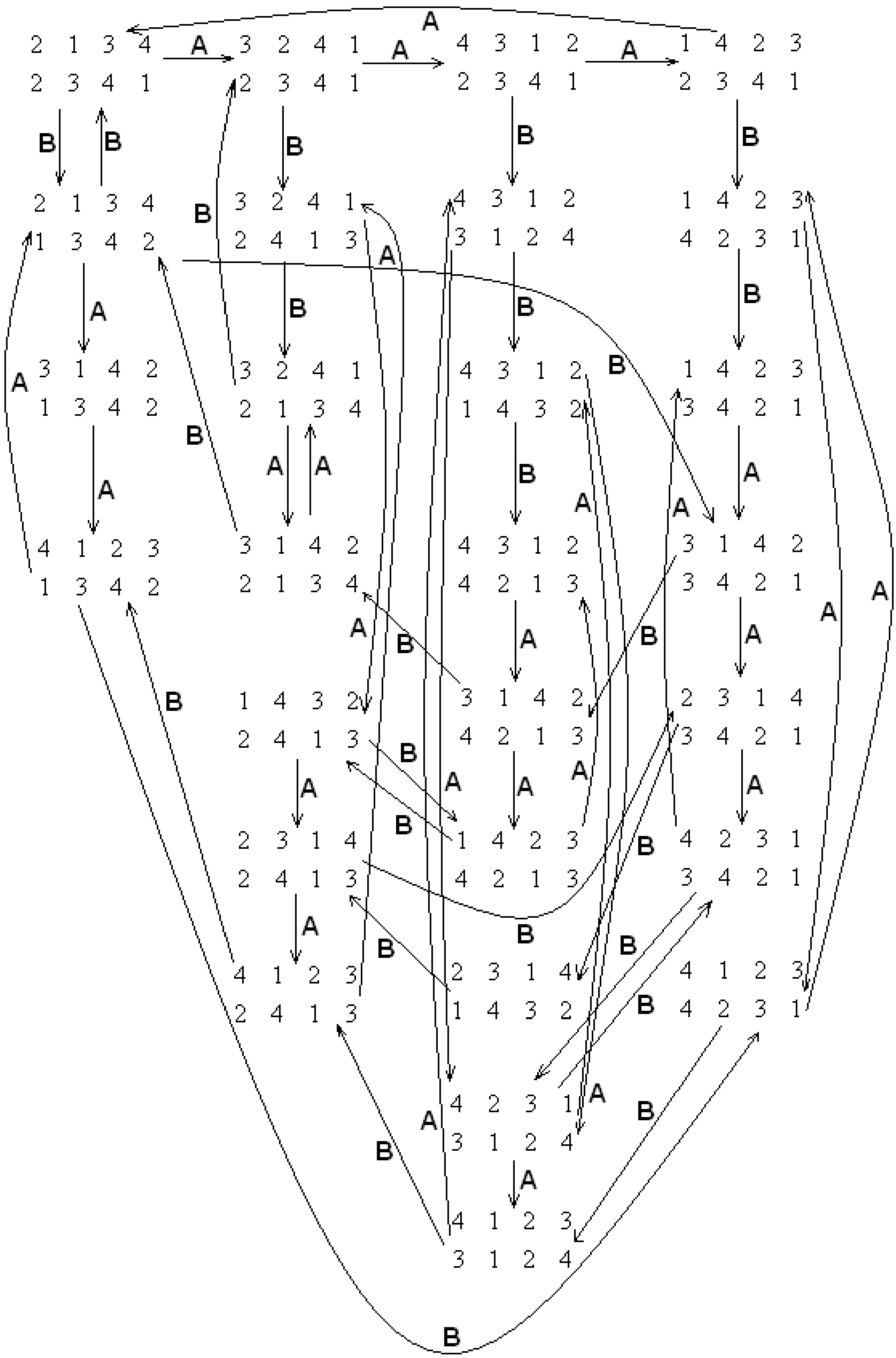


Fig. 1. The orbit of the pattern $\begin{matrix}2&1&3&4\\2&3&4&1\end{matrix}$.