## 5. Integer-valued Polynomials

An *integer-valued polynomial* $q(x)$ is a polynomial taking an integer value $q(n)$ for every positive integer $n$. Denote by $\mathbb{Q}_0[x]$ the set of all integer-valued polynomials with rational coefficients, that is

$$\mathbb{Q}_0[x]=\{q(x)\in\mathbb{Q}[x] \mid q(n)\in\mathbb{Z}, \forall n\in\mathbb{N}\}$$

Let $p$ be a prime number and let $\mathbb{Z}_p=\{0,1,\ldots,p-1\}$ be the set of residues modulo $p$.

1. Describe the set of integer-valued polynomials $q(x)\in\mathbb{Q}_0[x]$ with the property $q(n)\equiv 0 \; mod \; p$ for all $n\in\mathbb{N}$.
2. Let $q(x)$ be a polynomial from $\mathbb{Q}_0[x]$. Define whether the sequence $(q(n) \; mod \; p)_{n\in\mathbb{N}}$ is periodic and, if it is, find or estimate its period.
3. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{Z}_p$ is *realisable* if there exists an integer-valued polynomial $q(x)\in\mathbb{Q}_0[x]$ such that $q(n)\equiv a_n \; mod \; p$ for all $n\in\mathbb{N}$. Describe the set of realisable sequences.
4. Let $(a_n)_{n\in\mathbb{N}}$ be a realisable sequence. Describe the set of integer-valued polynomials $q(x)\in\mathbb{Q}_0[x]$ such that $q(n)\equiv a_n \; mod \; p$ for all $n\in\mathbb{N}$.
5. Describe the set $\mathbb{Q}_0[x]$.

Consider the problem for $n\in\mathbb{N}_0$. It's easy to see that we obtain equivalent problem by such a transition, because for each polynomial $q(x)$ from the original statement there will be a polynomial $q'(x)=q(x-1)$ for $n\in\mathbb{N}_0$ and contrary.

In the problem solution all calculations are made in the field $\mathbb{Z}_p$ by default.

Consider the polynomials $\binom{x}{k}=\frac{x(x-1)\ldots(x-k+1)}{k!}, k\in\mathbb{N}, \binom{x}{0}=1$. $\binom{x}{k}$ is an integer-valued polynomial $\forall k\in\mathbb{N}_0$. It's a well known fact, that if the polynomial $q(x)$ of degree $d$ takes integer values in the points $n, n+1, \ldots, n+d$ for some integer $n$, then it can be expressed uniquely in such way $q(x)=q_d\binom{x}{d}+q_{d-1}\binom{x}{d-1}+\ldots+q_1\binom{x}{1}+q_0\binom{x}{0}, q_i\in\mathbb{Z}$. It means $\left\{\binom{x}{k}, \forall k\in\mathbb{N}_0\right\}$ forms a basis for $\mathbb{Q}_0[x]$, so any integer-valued polynomial can be represented in a demanded kind. The proof of this fact can be found, for example, in [1].

Introduce the map $\varphi:\mathbb{Q}_0[x]\rightarrow\mathbb{Q}_0[x] \mid \varphi(q)=q(x+1)-q(x)$ (which is a linear operator in $\mathbb{Q}_0[x]$, and is the discrete analogue of derivative). It's easy to obtain that

$$\varphi(q_1+q_2)(x)=(q_1+q_2)(x+1)-(q_1+q_2)(x)=q_1(x+1)-q_1(x)+q_2(x+1)-q_2(x)=\varphi(q_1)(x)+\varphi(q_2)(x)$$

$$\forall x\in\mathbb{N}_0 \Rightarrow \varphi(q_1+q_2)=\varphi(q_1)+\varphi(q_2),$$

and therefore $\varphi(kq)=k\varphi(q), k\in\mathbb{N}_0$. $deg(\varphi^k(q))\leq deg(q)-k$, because $\varphi^n(q)(x+1)$ and $\varphi^n(q)(x)$ have the same coefficients at $x^{deg(\varphi^n(q))}$. Hence $\varphi^{deg(q)}(q)(x)$ is a constant, and $\varphi^{deg(q)+1}(q)(x)=0$.

Match the sequence $a(q)=(a_n)_{n\in\mathbb{N}_0} \mid a_n=q(n) \; mod \; p$ for every polynomial $q(x)\in\mathbb{Q}_0[x]$. Introduce the map $\psi$ for the sequences $\psi((a_n)_{n\in\mathbb{N}_0})=(a_1-a_0, a_2-a_1,\ldots)$. $\varphi^{deg(q)+1}(q)(x)=0$, hence $\psi^{deg(q)+1}(a(q))=(0, 0, 0,\ldots)$, because $\psi$ is the analogue of $\varphi$, for sequences. It also means that $\psi(a(q))=a(\varphi(q))$, and $\psi((a_n)+(b_n))=\psi((a_n))+\psi((b_n))$. We say that the sequence $(b_n)_{n\in\mathbb{N}_0}$ is *integrable* from $(a_n)_{n\in\mathbb{N}_0}$ if $\psi((b_n))=(a_n)$. We will get exactly $p$ sequences integrating the sequence $(a_n)_{n\in\mathbb{N}_0}$ in $\mathbb{Z}_p$, because for every fixed first element of the sequence we can calculate all other elements, and the first element can be any number from $\mathbb{Z}_p$. Only the sequence $(0, 0, 0,\ldots)$ integrates from itself (or we won't get $\psi^{deg(q)+1}(a(q))=(0, 0, 0,\ldots)$ ).

Now we will construct a tree which vertices are sequences (fig. 1). The root is $(0, 0, 0,...)$, it's children are $(1, 1, 1,...), (2, 2, 2,...), ..., (p-1, p-1, p-1,...)$, the children of every other sequence will be all the sequences, which are integrable from it. Introduce the following indexation — the index of $(0, 0, 0,...)$ is 0, the indexes of $(1, 1, 1,...), (2, 2, 2,...), ..., (p-1, p-1, p-1,...)$ are 1, 2, 3, ..., (p-1) respectively, and all the other sequences have the indexes of it's parent with the first element of the sequence added from right. We will also enumerate rows in a such way: the row containing $(0, 0, 0,...)$ and $(1, 1, 1,...),(2, 2, 2,...), ..., (p-1, p-1, p-1,...)$ has number 1, and the row containing the children of all elements from the previous row has the following number. So the $i^{\text{th}}$ row contains all the sequences, which give $(0, 0, 0,...)$ after applying the operation $\psi$ exactly $i$ times. Such a tree will contains all the realisable sequences, because each realisable sequence gives $(0, 0, 0,...)$ after a finite number of operations $\psi$.

We call $q(x)\in\mathbb{Q}_0[x]$ the base polynomial for realisable sequence $(a_n)_{n\in\mathbb{N}_0}$, if $(a_n)=a(q)$ and in decomposition $q(x)=q_d\binom{x}{d}+q_{d-1}\binom{x}{d-1}+...+q_1\binom{x}{1}+q_0\binom{x}{0},\ q_i\in\mathbb{Z}_p,\ d=deg(q)$. Such a polynomial always exists, because if we change some $q_i$ for $q_i'\equiv q_i \ mod\ p, q_i'\in\mathbb{Z}_p$ then the sequence $a(q)$ won't change. Every such $q(x)=q_d\binom{x}{d}+q_{d-1}\binom{x}{d-1}+...+q_1\binom{x}{1}+q_0\binom{x}{0},\ q_i\in\mathbb{Z}_p,$ is a base for $a(q)$.

Consider the sequences with the indexes 1, 10, 100, .... We arrange the elements of these sequences as on fig. 2. Enumerate these sequences from 0 consequently. It's easy to see that the numbers in triangle are the residues of numbers from Pascal triangle, because it is built same way as Pascal triangle. Hence the element in the $i^{\text{th}}$ row at the $k^{\text{th}}$ place is $\binom{i}{k} mod\ p$ (enumerating of rows and places starts from 0), and the elements of $k^{\text{th}}$ sequence are placed on the $k^{\text{th}}$ places, then $\binom{x}{k}$ is base polynomial for the $k^{\text{th}}$ sequence.

**Lemma 1:** The sequence $a\left(\binom{x}{k}\right)$ is periodic with minimal period $p^{j+1}$, where $p^j<k+1\le p^{j+1}$.

Suppose. $p^j<k+1\le p^{j+1}$ Show that $\binom{i}{k}\equiv\binom{i+p^{j+1}}{k} mod\ p, \forall i\in\mathbb{N}_0$. $\binom{i}{k}$ is the coefficient at $x^k$ in $(1+x)^i$, and $\binom{i+p^{j+1}}{k}$ is the coefficient at $x^k$ in $(1+x)^{i+p^{j+1}}$.
$(1+x)^{i+p^{j+1}}\equiv(1+x)^i(1+x)^{p^{j+1}}\equiv(1+x)^i(1+x^{p^{j+1}})\equiv(1+x)^i+(1+x)^i x^{p^{j+1}} mod\ p$. Here the coefficient at $x^k$ is equal to the coefficient in $(1+x)^i$, because in the second summand minimal degree is bigger than $k$, hence $\binom{i}{k}\equiv\binom{i+p^{j+1}}{k} mod\ p, \forall i\in\mathbb{N}_0$. Therefore the sequence $a\left(\binom{x}{k}\right)$ has the period $p^{j+1}$. S $p^j<k+1\le p^{j+1}$ incethen the first $p^j$ elements in $a\left(\binom{x}{k}\right)$ are equal to 0, and if $p^j$ is the period of this sequence, then this sequence is a sequence of zeroes, but it isn't. Therefore $p^j$ is not a period, so $p^{j+1}$ is the minimal period.

**Lemma 2:** $q(x)=i_0\binom{x}{k}+i_1\binom{x}{k-1}+...+i_{k-1}\binom{x}{1}+i_k\binom{x}{0}$ will be the base polynomial for the sequence with index $i_0 i_1 i_2 ... i_k$. I.e. all sequences in the tree are realisable.

**Lemma 3:** For each realisable sequence there exist only one base polynomial.

**Lemma 4:** All the sequences in the $d^{\text{th}}$ row of the tree, where $p^j<d\le p^{j+1}$, have the minimal period $p^{j+1}$.

For the sequences with indexes 0, 1, …, ($p$-1) lemma 2 is true. It's also true for the sequences with indexes 10, 100, 1000, …. Show that $a_{i_0 i_1 i_2 ... i_k}=i_0 a_{\underbrace{100..00}_{k\ zeroes}}+a_{i_1 i_2 ... i_k}, k\in\mathbb{N}_0$. For $k$=0 it's true, because $a_{i_0}=i_0 a_1$. Suppose we proved it for $k\le t$ show that it is true for $k$=$t$+1, i.e. $a_{i_0 i_1 i_2 ... i_{t+1}}=i_0 a_{\underbrace{100..00}_{t+1\ zeroes}}+a_{i_1 i_2 ... i_{t+1}}$. It is true because $\psi(i_0 a_{\underbrace{100..00}_{t+1\ zeroes}}+a_{i_1 i_2 ... i_{t+1}})=i_0\psi(a_{\underbrace{100..00}_{t+1\ zeroes}})+\psi(a_{i_1 i_2 ... i_{t+1}})=i_0 a_{\underbrace{100..00}_{t\ zeroes}}+a_{i_1 i_2 ... i_t}=a_{i_0 i_1 ... i_t}=\psi(a_{i_0 i_1 ... i_{t+1}})$, and they have the same first element. So $a_{i_0 i_1 i_2 \ldots i_k} = i_0 a_{\underbrace{100..00}_{k\,zeroes}} + i_1 a_{\underbrace{100..00}_{k-1\,zeroes}} + \ldots + i_{k-1} a_{10} + a_{i_0}$ , i.e. the sequences with the indexes 1, 10, 100,... form a basis for sequences in the tree. Thus for a sequence with index $i_0 i_1 i_2 \ldots i_k$ $q(x) = i_0 \binom{x}{k} + i_1 \binom{x}{k-1} + \ldots + i_{k-1} \binom{x}{1} + i_k \binom{x}{0}$ will be the base polynomial. Let's prove that there exist only one base polynomial for each realisable sequence. Let the sequence $(a_n)$ has two different base polynomials $q_1(x) = q_{1d_1} \binom{x}{d_1} + \ldots + q_{11} \binom{x}{1} + q_{10} \binom{x}{0}, d_1 = deg(q_1)$ and $q_2(x) = q_{2d_2} \binom{x}{d_2} + \ldots + q_{21} \binom{x}{1} + q_{20} \binom{x}{0}, d_2 = deg(q_2)$ . Then $a(q_1) = q_{1d_1} a_{\underbrace{100..00}_{d_1 zeroes}} + \ldots + q_{11} a_{10} + q_{10} a_1 = a_{q_{1d_1} \cdots q_{11} q_{10}}$ and $a(q_2) = q_{2d_2} a_{\underbrace{100..00}_{d_2 zeroes}} + \ldots + q_{21} a_{10} + q_{20} a_1 = a_{q_{2d_2} \cdots q_{21} q_{20}}$ , but $a_{q_{1d_1} \cdots q_{11} q_{10}} \neq a_{q_{2d_2} \cdots q_{21} q_{20}}$ because they have different indexes. Contradiction. Thus for a sequence with index $i_0 i_1 i_2 \ldots i_k$ the only base polynomial will be $q(x) = i_0 \binom{x}{k} + i_1 \binom{x}{k-1} + \ldots + i_{k-1} \binom{x}{1} + i_k \binom{x}{0}$ , because $a(q) = a_{i_0 i_1 i_2 \ldots i_k}$ and there cannot be another base polynomial.

Since $a_{i_0 i_1 i_2 \ldots i_k} = i_0 a_{\underbrace{100..00}_{k\,zeroes}} + i_1 a_{\underbrace{100..00}_{k-1\,zeroes}} + \ldots + i_{k-1} a_{10} + a_{i_0}$ , $p^{j+1}$ is period of $a_{i_0 i_1 i_2 \ldots i_k}$ , where $p^j < k \le p^{j+1}$ . $p + (p-1)p + (p-1)p^2 + \ldots + (p-1)p^{p^j - 1} = p^{p^j}$ sequences are placed in rows with numbers from 1 to $p^j$ and they all have period $p^j$ . Therefore all sequences with period $p^j$ are situated in rows with numbers from 1 to $p^j$ . So $p^j$ cannot be minimal period of $a_{i_0 i_1 i_2 \ldots i_k}$ . Hence $p^{j+1}$ is minimal period of $a_{i_0 i_1 i_2 \ldots i_k}$ , where $p^j < k \le p^{j+1}$ . In other words all the sequences in the $d^{th}$ row of the tree, where $p^j < d \le p^{j+1}$ , have the minimal period $p^{j+1}$ . Thus the sequence is realisable if and only if it has period $p^k, k \in \mathbb{N}_0$ .

Call $q_1(x)$ and $q_2(x)$ equal, if $a(q_1) = a(q_2)$ . It's easy to see that $q_1(x)$ and $q_2(x)$ are equal if and only if $q_{1i} \equiv q_{2i}\ mod\ p, \forall i \ge 0$ , where $q_{1i}$ and $q_{2i}$ are coefficients in the decomposition of $q_1(x)$ and $q_2(x)$ at $\binom{x}{i}$ , otherwise if we change coefficients in polynomials on their residues by modulo *p* we will get two base polynomials for one sequence, or one base polynomial for two different sequences. Contradiction. Hence if we know base polynomial for $(a_n)_{n \in \mathbb{N}_0}$ we can build all the polynomials $q(x) \in \mathbb{Q}_0[x]$ , such that $(a_n) = a(q)$ . Every polynomial $q(x)$ such that $a(q) = a_{i_0 i_1 \ldots i_k}$ is equal to base polynomial for this sequence, so it can expressed as

$$q(x) = i_0 \binom{x}{k} + i_1 \binom{x}{k-1} + \ldots + i_{k-1} \binom{x}{1} + i_k \binom{x}{0} + q_d \binom{x}{d} + q_{d-1} \binom{x}{d-1} + \ldots + q_1 \binom{k}{1} + q_0 \binom{k}{0}, p | q_j \ .$$

Thus we answered every question from the statement:

1. $q(x)=q_d\binom{x}{d}+q_{d-1}\binom{x}{d-1}+\ldots+q_1\binom{x}{1}+q_0\binom{x}{0},\ p|q_i$ .
2. Take $q'(x)\equiv q(x)$ , which is base, so the sequence $(a_n)$ is periodic with the minimal period $p^{k+1}$ , where $p^k\le deg(q')<p^{k+1}$ . We can take maximal *d*, such that $p\nmid q_d$ , instead of $deg(q')$ .
3. Sequence $(a_n)$ is realisable if and only if it has period $p^k, k\in\mathbb{N}_0$ .
4. For the sequence $(a_n)$ with index $i_0 i_1 i_2 \ldots i_k$ in our tree all polynomials $q(x)$ can be obtained by this formula:
$$q(x)=i_0\binom{x}{k}+i_1\binom{x}{k-1}+\ldots+i_{k-1}\binom{x}{1}+i_k\binom{x}{0}+q_k\binom{x}{k}+q_{k-1}\binom{x}{k-1}+\ldots+q_1\binom{k}{1}+q_0\binom{k}{0},\ p|q_j$$ .
Index of the sequence can be obtained such way: we add to the left of the string first element of sequence and then apply $\psi$ to the sequence. String will be index when we get sequence of zeroes.
5. $q(x)=q_d\binom{x}{d}+q_{d-1}\binom{x}{d-1}+\ldots+q_1\binom{x}{1}+q_0\binom{x}{0},\ q_i\in\mathbb{Z}$ .

If we return to the original statement $n\in\mathbb{N}$ , answers are so:

1. $q(x)=q_d\binom{x+1}{d}+q_{d-1}\binom{x+1}{d-1}+\ldots+q_1\binom{x+1}{1}+q_0\binom{x+1}{0},\ p|q_i$ .
2. Take $q'(x)\equiv q(x)$ , which is base, so the sequence $(a_n)$ is periodic with the minimal period $p^{k+1}$ , where $p^k\le deg(q')<p^{k+1}$ . We can take maximal *d*, such that $p\nmid q_d$ , instead of $deg(q')$ .
3. Sequence $(a_n)$ is realisable if and only if it has period $p^k, k\in\mathbb{N}_0$ .
4. For the sequence $(a_n)$ with index $i_0 i_1 i_2 \ldots i_k$ in our tree all polynomials $q(x)$ can be obtained by this formula:
$$q(x)=i_0\binom{x+1}{k}+i_1\binom{x+1}{k-1}+\ldots+i_k\binom{x+1}{0}+q_k\binom{x+1}{k}+q_{k-1}\binom{x+1}{k-1}+\ldots+q_1\binom{x+1}{1}+q_0\binom{x+1}{0},\ p|q_j$$
Index of the sequence can be obtained such way: we add to the left of the string first element of sequence and then apply $\psi$ to the sequence. String will be index when we get sequence of zeroes.
5. $q(x)=q_d\binom{x+1}{d}+q_{d-1}\binom{x+1}{d-1}+\ldots+q_1\binom{x+1}{1}+q_0\binom{x+1}{0},\ q_i\in\mathbb{Z}$

Prospects of further research: consider polynomials of many variables.

**References and Articles:**


1. V. Prasolov «Polynomials», MCCME, 1999.
2. V. Arnold «Complexity of finite sequences of zeroes and ones and geometry of finite spaces and functions», Functional Analysis and Other Mathematics, 2007.

## Appendix

If we prove $\varphi^{deg(q)+1}(q)(x)=0$ the other way we can obtain the following interesting property:

$$(-1)^0 C_{d+1}^0 x^i+(-1)^1 C_{d+1}^1 (x+1)^i+...+(-1)^d C_{d+1}^d (x+d)^i+(-1)^{d+1} C_{d+1}^{d+1} (x+d+1)^i=0, 0\le i\le d, \forall x\in\mathbb{N} .$$

Calculate the coefficient at $q_i$ in $\varphi^{deg(q)+1}(q)(x)$ , where $q(x)=q_d x^d+q_{d-1} x^{d-1}+...+q_1 x+q_0$ . It is $(-1)^0 C_{d+1}^0 x^i+(-1)^1 C_{d+1}^1 (x+1)^i+...+(-1)^d C_{d+1}^d (x+d)^i+(-1)^{d+1} C_{d+1}^{d+1} (x+d+1)^i$ . Let's prove, that it is equal 0 $\forall x\in\mathbb{N}$ and $0\le i\le d$ .

If $d$=0 then $i$=0: $C_1^0 x^0-C_1^1 (x+1)^0=0$ — it's true $\forall x\in\mathbb{N}$ .
Suppose we proved the statement for $d$=$k$ and for $0\le i\le k$ :
$$(-1)^0 C_{k+1}^0 x^i+(-1)^1 C_{k+1}^1 (x+1)^i+...+(-1)^k C_{k+1}^k (x+k)^i+(-1)^{k+1} C_{k+1}^{k+1} (x+k+1)^i=0 .$$

Then let's prove it for $d$=$k$+1 and $0\le i\le k+1$ .
Substracting equalities
$$(-1)^0 C_{k+1}^0 x^i+(-1)^1 C_{k+1}^1 (x+1)^i+...+(-1)^k C_{k+1}^k (x+k)^i+(-1)^{k+1} C_{k+1}^{k+1} (x+k+1)^i=0$$ and
$$(-1)^0 C_{k+1}^0 (x+1)^i+(-1)^1 C_{k+1}^1 (x+2)^i+...+(-1)^k C_{k+1}^k (x+k+1)^i+(-1)^{k+1} C_{k+1}^{k+1} (x+k+2)^i=0 .$$
We obtain:
$$C_{k+1}^0 x^i+(-1)^1 (C_{k+1}^1+C_{k+1}^0)(x+1)^i+...+(-1)^{k+1} (C_{k+1}^{k+1}+C_{k+1}^k)(x+k+1)^i+(-1)^{k+2} C_{k+1}^{k+1} (x+k+2)^i=0$$
$$(-1)^0 C_{k+2}^0 x^i+(-1)^1 C_{k+2}^1 (x+1)^i+...+(-1)^{k+1} C_{k+2}^{k+1} (x+k+1)^i+(-1)^{k+2} C_{k+2}^{k+2} (x+k+2)^i=0$$
for all $0\le i\le k$ .
Now show that it's true for $i$=$k$+1:
$$(-1)^0 C_{k+2}^0 x^{k+1}+(-1)^1 C_{k+2}^1 (x+1)^{k+1}+...+(-1)^{k+1} C_{k+2}^{k+1} (x+k+1)^{k+1}+(-1)^{k+2} C_{k+2}^{k+2} (x+k+2)^{k+1} .$$
Change $x$ for $t$+1, where $t\ge 0$ :
$$(-1)^0 C_{k+2}^0 (t+1)^{k+1}+(-1)^1 C_{k+2}^1 (t+2)^{k+1}+...+(-1)^{k+1} C_{k+2}^{k+1} (t+k+2)^{k+1}+(-1)^{k+2} C_{k+2}^{k+2} (t+k+3)^{k+1} .$$
Calculating coefficient at $t^j$ , for $0\le j\le k$ , obtain:
$$(-1)^0 C_{k+2}^0 C_{k+1}^j 1^j+(-1)^1 C_{k+2}^1 C_{k+1}^j 2^j+...+(-1)^{k+1} C_{k+2}^{k+1} C_{k+1}^j (k+2)^j+(-1)^{k+2} C_{k+2}^{k+2} C_{k+1}^j (k+3)^j$$
$$C_{k+1}^j ((-1)^0 C_{k+2}^0 1^j+(-1)^1 C_{k+2}^1 2^j+...+(-1)^{k+1} C_{k+2}^{k+1} (k+2)^j+(-1)^{k+2} C_{k+2}^{k+2} (k+3)^j)=0 .$$
And the coefficient at $t^{k+1}$ is $C_{k+2}^0-C_{k+2}^1+...+(-1)^{k+1} C_{k+2}^{k+1}+(-1)^{k+2} C_{k+2}^{k+2}=0$ .
Hence for all $t\ge 0$
$$(-1)^0 C_{k+2}^0 (t+1)^{k+1}+(-1)^1 C_{k+2}^1 (t+2)^{k+1}+...+(-1)^{k+1} C_{k+2}^{k+1} (t+k+2)^{k+1}+(-1)^{k+2} C_{k+2}^{k+2} (t+k+3)^{k+1}=0 .$$

Therefore
$$(-1)^0 C_{d+1}^0 x^i+(-1)^1 C_{d+1}^1 (x+1)^i+...+(-1)^d C_{d+1}^d (x+d)^i+(-1)^{d+1} C_{d+1}^{d+1} (x+d+1)^i=0, \forall x\in\mathbb{N}, 0\le i\le d .$$

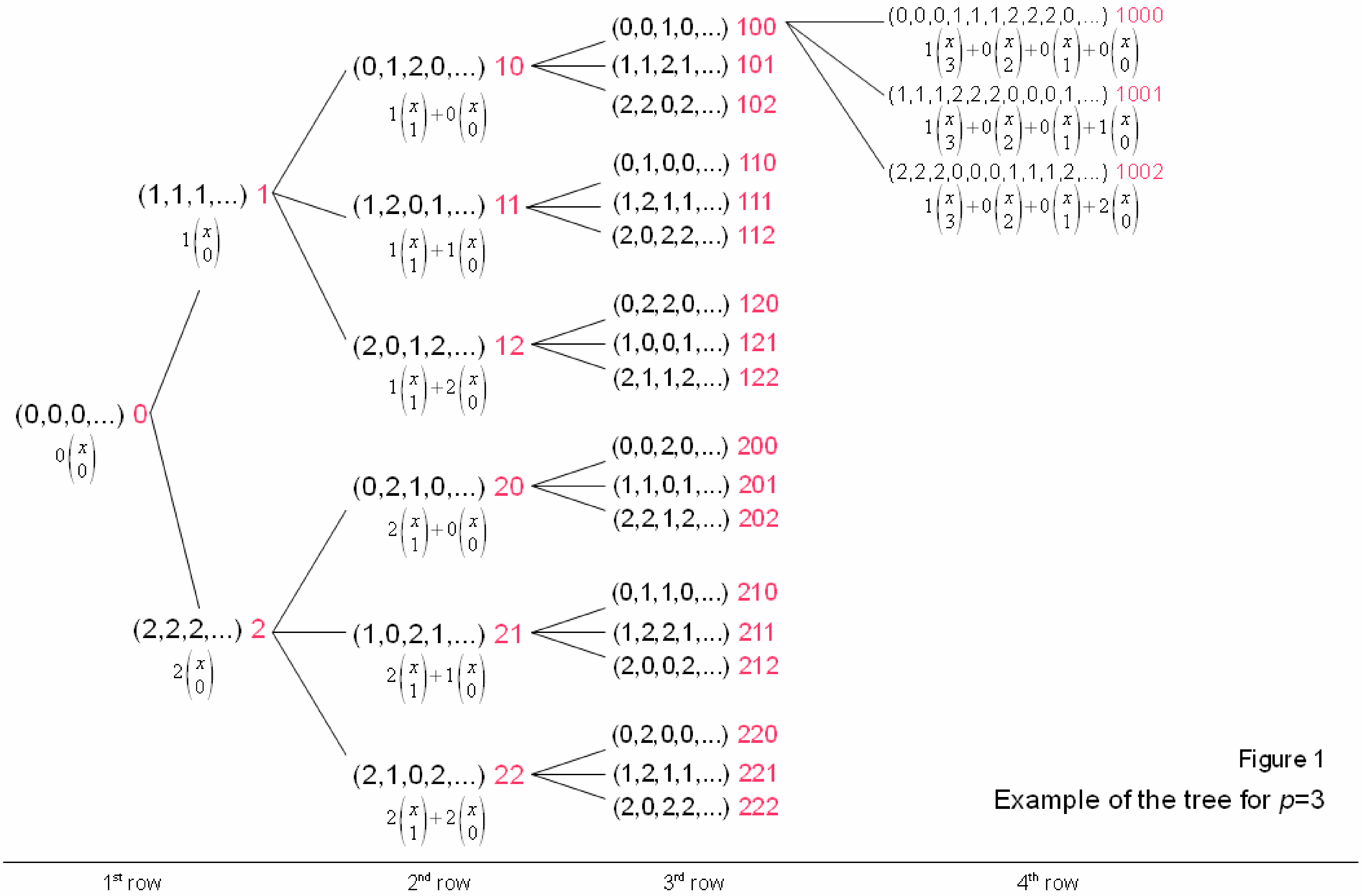
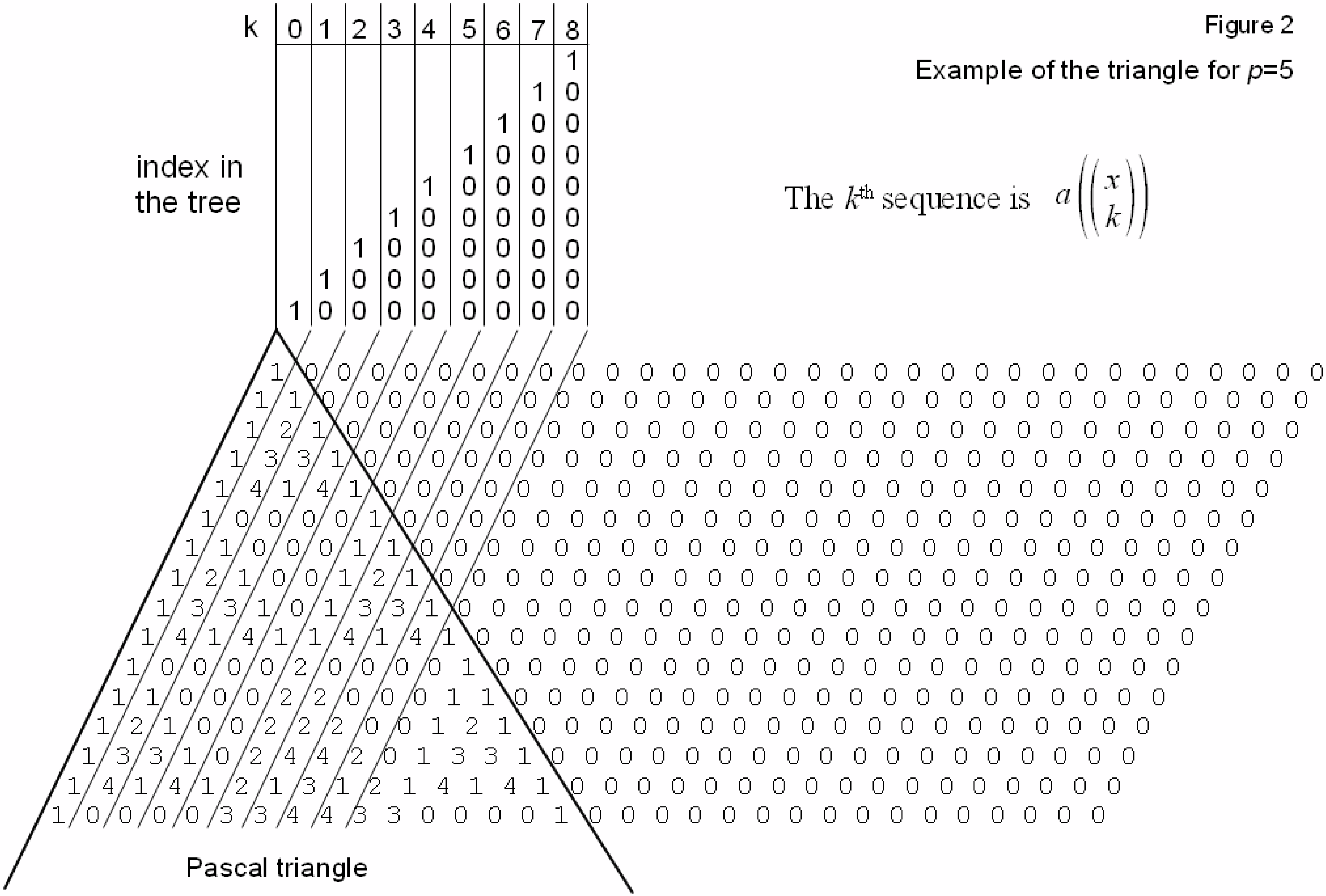



Figure 1

Example of the tree for $p$=3

Figure 2

Example of the triangle for $p$=5




The $k^{th}$ sequence is $a\left(\binom{x}{k}\right)$