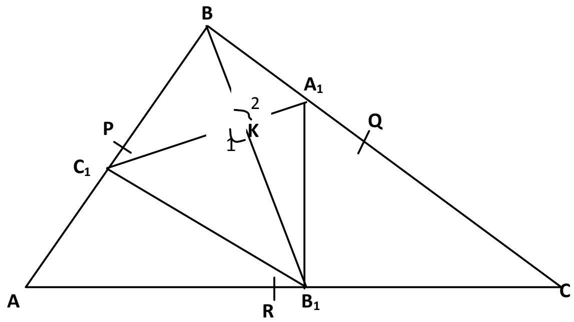
№4. Minimality of Inscribed Polygons

№1. A



Can the area of $\Delta A_1B_1C_1$ be less than area of any of three triangles $\Delta AC_1B_1, \Delta B_1A_1C, \Delta A_1C_1B$?

Answer: No, it can't. Let's prove it.

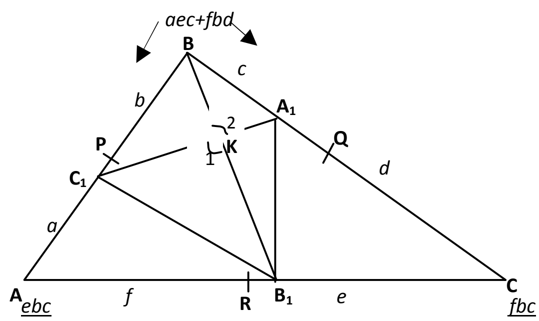
Suppose $\Delta A_1B_1C_1$ is inscribed in ΔABC, and $S_{\Delta A_1B_1C_1} < \min\{S_{\Delta A_1BC_1}, S_{\Delta AC_1B_1}, S_{\Delta A_1CB_1}\}$. Denote $K = BB_1 \cap A_1C_1$. Denote the height dropped from the vertex B in ΔA_1C_1B by h_2 and the height dropped from the vertex B_1 in $\Delta A_1B_1C_1$ by h_1. Notice that $S_{\Delta A_1B_1C_1} < S_{\Delta A_1BC_1} \Leftrightarrow h_1 \cdot A_1C_1 < h_2 \cdot A_1C_1 \Leftrightarrow h_1 < h_2 \Leftrightarrow B_1K \cdot \sin\angle 1 < BK \cdot \sin\angle 2 \Leftrightarrow B_1K < BK$. ($\angle 1 = \angle 2$, because they are vertical). Similarly, it can be shown that $M = CC_1 \cap A_1B_1 \Rightarrow C_1M < CM,\ N = AA_1 \cap B_1C_1 \Rightarrow A_1N < AN$.

Denote the middle points of AB, BC, AC by P, Q, R respectively. Now notice that

$\begin{cases} C_1 \in [PB] \\ A_1 \in [BQ] \end{cases} \Rightarrow BK \le B_1K$, (because A_1C_1 lies above the middle line).

$$\begin{cases} C_1 \in [AP] \\ A_1 \in [QC] \end{cases}, \left[\begin{array}{l} B_1 \in [AR] \\ B_1 \in [RC] \end{array}\right. \Rightarrow \left[\begin{array}{l} CM \le C_1M \\ AN \le A_1N \end{array}\right.$$

Without loss of generality suppose $C_1 \in (AP), A_1 \in (BQ), B_1 \in (CR)$. Denote $AC_1 = a, C_1B = b, BA_1 = c, A_1C = d, CB_1 = e, B_1A = f$. then $b > a, f > e, d > c$.



Put masses into vertices of the triangle the way like on the figure above. Thus we obtain that the mass center of ΔABC lies on A_1C_1 and on BB_1. I.e. K is the mass center.

Therefore $BK > KB_1 \Leftrightarrow aec + fbd < ebc + fbc$.

Put masses in vertices of ΔABC the same way so that the mass center is in the point M or N.

Then we have $CM > MC_1 \Leftrightarrow aec + fbd < bed + ade$, $AN > NA_1 \Leftrightarrow aec + fbd < adf + acf$.

So the assumption that $S_{\Delta A_1B_1C_1} < \min\{S_{\Delta A_1BC_1}, S_{\Delta AC_1B_1}, S_{\Delta A_1CB_1}\}$ is equivalent to the next system

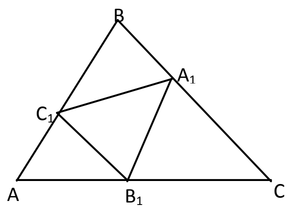
$$\begin{cases} aec + fbd < ebc + fbc \\ aec + fbd < bed + ade \quad (1) \\ aec + fbd < adf + acf \end{cases} \Rightarrow \quad 3aec + 3fbd < ebc + fbc + bed + ade + adf + acf \Leftrightarrow$$

$\Leftrightarrow 2(ac - bd)(e - f) < (e + f)(b - a)(c - d)$.(*)

But: $\begin{cases} b > a \\ f > e \\ d > c \end{cases} \Rightarrow \begin{cases} ac - bd < 0 \\ e - f < 0 \\ b - a > 0 \\ c - d < 0 \end{cases}$, consequently (*) is wrong, so at least one inequality of the system (1) doesn't hold, so our basic assumption is wrong.

Then $S_{\Delta A_1B_1C_1} \ge \min\{S_{\Delta A_1BC_1}, S_{\Delta AC_1B_1}, S_{\Delta A_1CB_1}\}$ for any inscribed $\Delta A_1B_1C_1$.

№1. B.

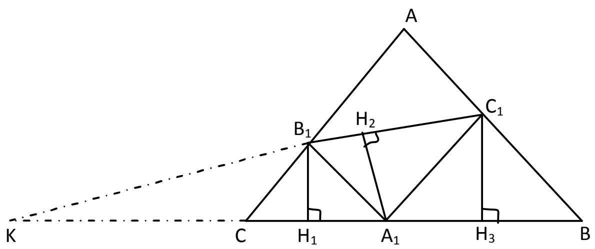


1) Can one of three angles of $\Delta A_1B_1C_1$ be smaller than any angle of $\Delta AC_1B_1, \Delta B_1A_1C, \Delta A_1C_1B$?

Answer: Yes, it can. Show that for any ΔABC there exists such $\Delta A_1B_1C_1$.

Denote a middle point of the side BC by A_1. Move the points B_1 and C_1 towards the vertex A with the condition $B_1C_1 \parallel BC$. We see that the lengths of AB_1 and AC_1 are bent on 0. Since $\angle B_1A_1C \to \angle AA_1C$, $\angle C_1A_1B \to \angle AA_1B$, so $\angle C_1A_1B_1 = \pi - \angle B_1A_1C - \angle C_1A_1B \to 0$ (*). Now we'll show that $\angle C_1A_1B_1$ can be smaller than any angle of $\Delta AC_1B_1, \Delta B_1A_1C, \Delta A_1C_1B$.

Indeed, angles $\angle C_1AB_1, \angle AC_1B_1 = \angle ABC, \angle AB_1C_1 = \angle ACB\ (B_1C_1 \parallel BC)$ are constants. Under the conditions of (*), we can obtain the angle which is smaller then each of them. $\angle C_1A_1B$ and $\angle B_1A_1C$ are bent on a constant positive value of $\angle AA_1B$ and $\angle AA_1C$ respectively because we can move C_1 and B_1 to A as close as we need to. Also $\angle CB_1A_1 \to \angle CAA_1$ and $\angle BC_1A_1 \to \angle BAA_1$. So we can choose such a position of C_1 and B_1 that $\angle C_1A_1B_1$ will be the smallest among angles of $\Delta A_1B_1C_1$, $\Delta AC_1B_1, \Delta B_1A_1C, \Delta A_1C_1B$.

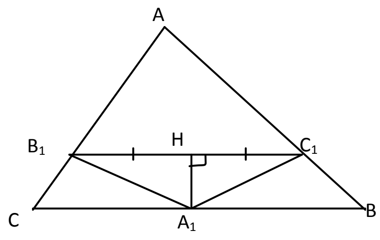


2) Can a height of $\Delta A_1B_1C_1$ be smaller than any height of $\Delta AC_1B_1, \Delta B_1A_1C, \Delta A_1C_1B$?

Answer: No, it can't. Let's prove it.

Suppose $\Delta A_1B_1C_1$ is inscribed in ΔABC. Consider its smallest height. Since $S_\Delta = \frac{1}{2} h_a \cdot a$ then the smallest height of triangle is dropped on the greatest edge. The greatest edge of triangle lies opposite the greatest angle. W. l. o. g. suppose $\angle B_1A_1C_1$ is the greatest angle of $\Delta A_1B_1C_1$. Then the height A_1H_2 isn't longer than the other heights of $\Delta A_1B_1C_1$. Also $\angle B_1A_1C_1$ can't be an acute angle in a not acute-angled triangle because $\angle B_1A_1C_1$ is the greatest. So H_2 lies strictly within the segment B_1C_1. Let's draw heights B_1H_1 and C_1H_3 in ΔB_1A_1C and ΔA_1C_1B. There are 2 cases:

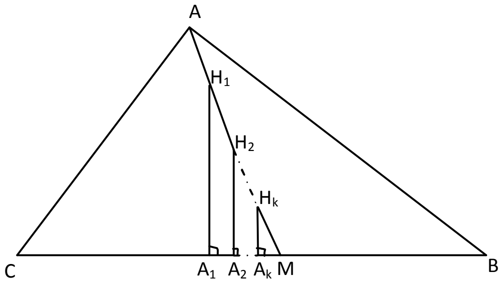
1. $B_1C_1 \nparallel BC$. Then suppose $K = B_1C_1 \cap BC$. Assume K lies on the elongation of BC behind the point C. Notice that $\Delta KB_1H_1 \sim \Delta KH_2A_1$ by 2 angles. And since $KH_2 > KB_1 > KH_1$ we have $\frac{A_1H_2}{B_1H_1} > 1$. The case when K lies on the elongation of BC behind the point B can be considered in the same way ■
2. $B_1C_1 \parallel BC$. Then $A_1H_2 = B_1H_1$ as the distanse between lines B_1C_1 and BC. ■



3) Can a median/bisector/zhergonian of $\Delta A_1B_1C_1$ be smaller than any median/bisector/zhergonian of $\Delta AC_1B_1, \Delta B_1A_1C, \Delta A_1C_1B$?

Answer: Yes, it can. Let's show that for any ΔABC there exists such $\Delta A_1B_1C_1$.

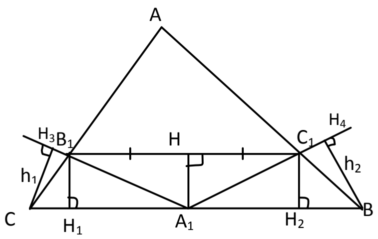
W. l. o. g. let $\angle BAC$ be the greatest angle of ΔABC. Consequently the angles $\angle ABC, \angle BCA$ are acute (*). Move points B_1 and C_1 toward the vertices C and B respectively along the sides of ΔABC with the condition $B_1C_1 \parallel BC$. Let A_1 be the point of intersection of the middle perpendicular to the segment B_1C_1 with the side BC. It means that $A_1B_1 = A_1C_1$. Thus the segment A_1H is a height, median, bisector and zhergonian in $\Delta A_1B_1C_1$.

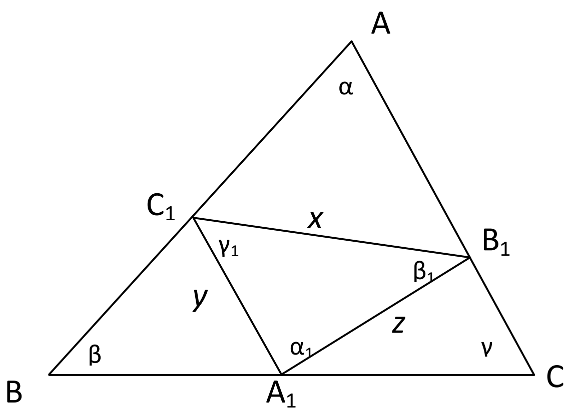


Let us prove that the coordinates of the point A_1 are bent on the coordinates of the middle point of BC as we move points B_1 and C_1 to vertexes C and B respectively. Indeed middle points H) of all such segments $B_1C_1 \parallel BC$ lie on the median AM of ΔABC. This yields that the case $AB = AC$ is obvious. Consider the case $AB \neq AC$. The length segment A_1H is bent on 0 as we move points B_1 and C_1 to vertexes C and B then the length (like the distance between B_1C_1 and BC). All ΔMH_kA_{1k} are similar, so we have $|MA_1| \longrightarrow 0$. ■

Now let's prove that we can choose such points B_1 and C_1 on sides AC and AB respectively that the length of A_1H is less than the length of any median, bisector and zhergonian of $\Delta AC_1B_1, \Delta B_1A_1C, \Delta A_1C_1B$.

If we approach points B_1 and C_1 to vertices C and B respectively then the lengths of medians, bisectors, zhergonians in ΔAC_1B_1 are bent on constant values of lengths of this chievians in ΔABC. We can make the segment A_1H as short as we want (it follows from above). Consider the chievians leaving from the vertex A_1 of $\Delta B_1A_1C, \Delta A_1C_1B$. Lengths of this chievians are not bent on 0 because the coordinates of the point A_1 are bent on the coordinates of the middle point of BC as we move points B_1 and C_1 to vertexes C and B, we can say the same about A_1C_1, A_1B_1 (they are greater than distances between A_1 and sides AC and AB which are bent on distances between M and them). Also we can make $A_1C_1 \neq BC_1, A_1B_1 \neq CB_1$ because $|BC_1| \longrightarrow 0, |CB_1| \longrightarrow 0$. Then notice that the heights B_1H_1 and C_1H_2 which are equal to A_1H aren't medians, bisectors or zhergonians in ΔB_1A_1C and ΔA_1C_1B. It means that lengths of these heights are less than these chievians. Now consider the similar chievians from vertexes C and B of ΔB_1A_1C and ΔA_1C_1B respectively. Notice that none of them is smaller than the height dropped from the correspondent vertex. Also notice that if $\angle CB_1A_1 \geq 90°, \angle A_1C_1B \geq 90°$ then the feet of heights h_1 and h_2 is situated either at vertexes B_1 and C_1 or on elongations of the sides of the triangle behind B_1 and C_1 respectively (under the conditions of (*)). So we have $CH_3 > B_1H_1$ and $BH_4 > C_1H_2$ because $\Delta CH_3A_1 \sim \Delta H_1B_1A_1$, $\Delta H_4BA_1 \sim \Delta A_1C_1H_2$ and $CA_1 > A_1B_1$, $BA_1 > A_1C_1$ (the greatest angle is opposite the greatest edge). Lengths of chievians which are drawn from vertexes C and B are greater than the length of A_1H because $B_1H_1 = C_1H_2 = A_1H$. Finally, we need to show that we can obtain $\angle CB_1A_1 \geq 90°, \angle A_1C_1B \geq 90°$. $|BC_1| \longrightarrow 0, |CB_1| \longrightarrow 0$, when we move B_1 and C_1 to vertexes C and B respectively. Since $\angle C_1A_1B = \arcsin \frac{C_1B \cdot \sin\angle ABC}{A_1C_1}$, $\angle B_1A_1C = \arcsin \frac{B_1C \cdot \sin\angle ACB}{A_1B_1}$ and A_1C_1, A_1B_1 are bent on *MB* and *MC* respectively, then we see that $\angle C_1A_1B$ and $\angle B_1A_1C$ are bent on 0. Under the conditions of (*) we can choose such positions of points B_1 and C_1 that $\angle CB_1A_1 \geq 90°, \angle A_1C_1B \geq 90°$. ■.





4) Can the radius of the circumscribed circle of $\Delta A_1B_1C_1$ be smaller than radius of the circumscribed circle of any of three triangles $\Delta AC_1B_1, \Delta B_1A_1C, \Delta A_1C_1B$?

Answer: if ΔABC is acute-angled or right, then it can't. If ΔABC is obtuse-angled, then it can.

Denote radii of the circumscribed circle of $\Delta A_1B_1C_1, \Delta AC_1B_1, \Delta A_1C_1B, \Delta B_1A_1C$ by R_0, R_1, R_2, R_3 respectively.

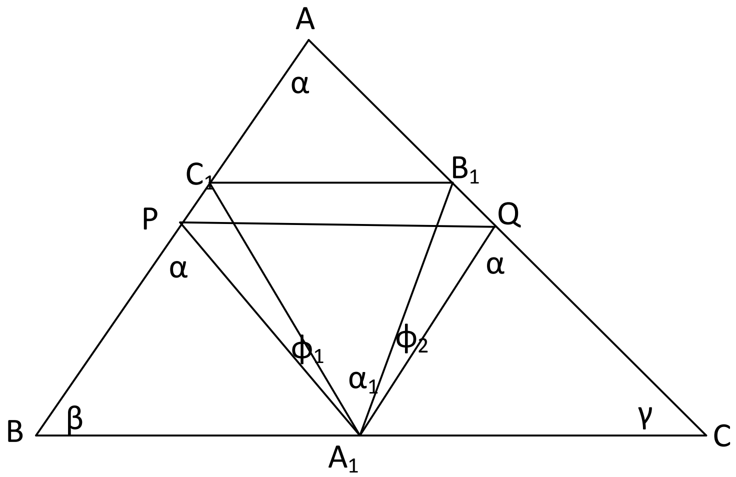
So we have:

$$\begin{cases} R_1 > R_0 \\ R_2 > R_0 \\ R_3 > R_0 \end{cases} \Leftrightarrow \begin{cases} \frac{x}{\sin\alpha} > \frac{x}{\sin\alpha_1} \\ \frac{y}{\sin\beta} > \frac{y}{\sin\beta_1} \\ \frac{z}{\sin\gamma} > \frac{z}{\sin\gamma_1} \end{cases} \Leftrightarrow \begin{cases} \sin\alpha_1 > \sin\alpha \\ \sin\beta_1 > \sin\beta \\ \sin\gamma_1 > \sin\gamma \end{cases} (1)$$

1) Suppose $\alpha, \beta, \gamma \le \frac{\pi}{2}$, then if (1) is true, then $\begin{cases} \alpha_1 > \alpha \\ \beta_1 > \beta \\ \gamma_1 > \gamma \end{cases}$ is also true, and so

$$\pi = \alpha_1 + \beta_1 + \gamma_1 > \alpha + \beta + \gamma = \pi.$$

So the original statement is wrong.



2) Let $\alpha > \frac{\pi}{2}$. So $\beta, \gamma < \frac{\pi}{2}$. Let's show that there exists such $\Delta A_1B_1C_1$ that (1) is true. Let P, Q, A_1 be middle points of sides AB, AC, BC respectively. Notice that the angles of ΔA_1QP are respectively equal to the angles of ΔABC. Consider $B_1 \in AQ, C_1 \in AP$ such that $B_1C_1 \parallel PQ$. Thus $\angle B_1C_1A_1 = \angle C_1A_1B = \gamma_1 = \gamma + \varphi_1$ and $\angle C_1B_1A_1 = \angle B_1A_1C = \beta_1 = \beta + \varphi_2$ (see the figure). Also $\angle B_1A_1C_1 = \alpha_1 = \alpha - \varphi_1 - \varphi_2$. Notice, that if $\alpha > \alpha_1 \ge \frac{\pi}{2}$, $\beta < \beta_1 \le \frac{\pi}{2}$, $\gamma < \gamma_1 \le \frac{\pi}{2}$ then (1) is true. We can make $\alpha > \alpha_1 \ge \frac{\pi}{2}$, $\beta < \beta_1 \le \frac{\pi}{2}$, $\gamma < \gamma_1 \le \frac{\pi}{2}$ if φ_1, φ_2 can be infinitesimal. This Fact is true because we can move points B_1, C_1 to the points Q, P respectively as close as we want. So $B_1Q \to 0$, $C_1P \to 0$ with approaching of B_1, C_1 and therefore $\angle C_1A_1P = \varphi_1 = arcsin\frac{C_1P\cdot\sin(\pi-\alpha)}{A_1C_1} \to 0$, $\angle C_1B_1Q = \varphi_2 = arcsin\frac{B_1Q\cdot\sin(\pi-\alpha)}{A_1B_1} \to 0$, because A_1C_1, A_1B_1 are not less than distances between A_1 and lines AB, AC respectively. ■

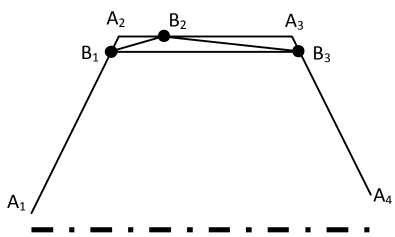
№2.

1) A convex polygon $B_1B_2 \dots B_m$ is inscribed in a convex polygon $A_1A_2 \dots A_n$, $3 \le m \le n$, so that $A_1A_2 \dots A_n$ is divided into $m+1$ parts. Can $B_1B_2 \dots B_m$ have smaller diagonal than any diagonal of any such part? (if this part is not $B_1B_2 \dots B_m$ then let us call it an outside polygon)

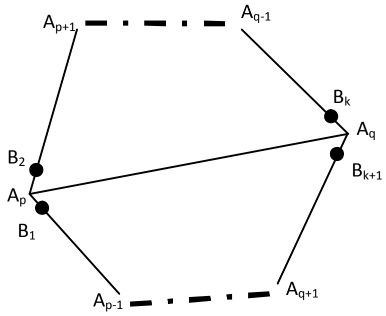
Answer: Yes, it can.

Let $3 < m < n$, because inscribed polygon and at least one outside polygon must have at least one diagonal.

Solve this item of our problem for the case, when there exists the only least element in the set of all diagonals and edges of $A_1A_2 \dots A_n$.



Consider the case when this is the edge of $A_1A_2 \dots A_n$. W. l. o. g. let A_2A_3 be that edge. Now determine the positions of three vertexes of the polygon $B_1B_2 \dots B_m$. Suppose B_1 lies on A_1A_2, B_2 lie on A_2A_3, B_3 lie on A_3A_4. Move points B_1 and B_3 to vertexes A_2 and A_3 respectively with the condition $A_2A_3 \parallel B_1B_3$. Notice that the length of the segmen B_1B_3 will be bent on the length of A_2A_3. So under the conditions of minimality of A_2A_3 we can choose such positions of points B_1 and B_3 that the length of B_1B_3 will be less than the length of any diagonal and any edge of $A_1A_2 \dots A_n$ except A_2A_3. We can choose position of the vertex B_2 on A_2A_3 arbitrarily and then B_1B_3 is a diagonal of $B_1B_2 \dots B_m$. Place other $m-3$ vertexes of the inscribed polygon on other edges of $A_1A_2 \dots A_n$ and move each of them to one of two the endpoints of a correspondent edge. Prove that there is no outside polygon that has a smaller diagonal than B_1B_3. Consider an arbitrary vertex $B_i, i \ne 2$. Let it be on the edge A_jA_{j+1}. And we move it, w. l. o. g., to the point A_{j+1}. This means that lengths of segments $B_iA_k (k \ne j, j+1)$ are bent on lengths of segments $A_{j+1}A_k$. Consider outside polygons which have the vertex B_i. Lengths of their diagonals are bent on lengths of correspondent segments (edges or diagonals) in $A_1A_2 \dots A_n$. I.e. lengths of diagonal in outside polygons can differ from those segments infinitesimally. So we can choose such position of B_i that any diagonal of an outside polygon with the endpoints in the point B_i will be greater than B_1B_3.

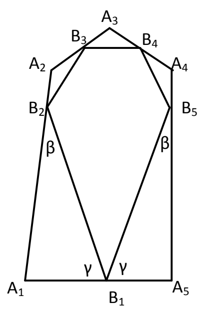


Now consider the case when the least element in the set of all diagonals and edges of $A_1A_2 \dots A_n$ is a diagonal. Let A_pA_q be this diagonal. Then we'll place 4 vertexes (let B_1, B_2, B_k, B_{k+1} be them) of $B_1B_2 \dots B_m$ on edges of $A_1A_2 \dots A_n$, as one can see on the figure. And we'll move B_1 and B_2 to A_p, B_k and B_{k+1} to A_q. Notice that lengths of segments B_1B_k, B_1B_{k+1}, B_2B_k, B_2B_{k+1} are bent on the length of A_pA_q. So under the conditions of minimality of we can choose such positions of B_1, B_2, B_k, B_{k+1} that the length of any of the segments B_1B_k, B_1B_{k+1}, B_2B_k, B_2B_{k+1} will be less than the length of any diagonal and edge of $A_1A_2 \dots A_n$ except A_pA_q. We will place other $m-3$ vertexes of an inscribed polygon on other edges of $A_1A_2 \dots A_n$ and move each of them to one of two vertexes, which are the endpoints of a correspondent edge. Prove that there is no outside polygon that have smaller diagonal than any of the segments B_1B_k, B_1B_{k+1}, B_2B_k, B_2B_{k+1}. Consider an arbitrary vertex B_i. Let it be on the edge A_uA_v And we move it, w. l. o. g., to the point A_v. This means that lengths of segments $B_iA_k\ (k \neq u, v)$ are bent on lengths of segments A_vA_k. Consider outside polygons which have the vertex B_i. Lengths of their diagonals are bent on lengths of correspondent segments (edges or diagonals) in $A_1A_2 \dots A_n$. I.e. lengths of diagonal in outside polygons can differ from those segments infinitesimally. So we can choose such position of B_i that any diagonal of an outside polygon with the end in the poin B_i will be greater than any of the segments B_1B_k, B_1B_{k+1}, B_2B_k, B_2B_{k+1}. Therefore the initial proposition is proved.

2) A convex polygon $B_1B_2 \dots B_m$ is inscribed in a convex polygon $A_1A_2 \dots A_n$, $3 \le m \le n$, so that $A_1A_2 \dots A_n$ is divide into $m+1$ parts. Can $B_1B_2 \dots B_m$ have smaller angle than any angle of any such part? (if this part is not $B_1B_2 \dots B_m$ then let us call it an outside polygon)

Answer: Yes, it can.

We shall show that for any m and n there exists such polygons $B_1B_2 \dots B_m$ and $A_1A_2 \dots A_n$.



Construct a convex polygon $B_1B_2 \dots B_m$ such that $\angle B_2B_1B_m = \frac{\pi(m-2)}{m} - \alpha$ and other its angles are equal and value of each of them is $\frac{\pi(m-2)}{m} + \frac{\alpha}{m-1}$, where $0 \le \alpha < \frac{\pi(m-2)}{m}$ is a parameter. Then draw the line l_i through each vertex B_i, so that l_i is not a bisector of $\angle B_i$ and the angle between l_i and the side $B_{i-1}B_i$ is equal to the angle between l_i. The side B_iB_{i+1} and equal to $\beta = \frac{\pi}{2} - \frac{\pi(m-2)}{2m} - \frac{\alpha}{2(m-1)}$ for $i = \overline{2, m}$; or $\gamma = \frac{\pi}{2} - \frac{\pi(m-2)}{2m} + \frac{\alpha}{2}$ for $i = 1$. Notice that lines l_i and l_{i+1}, $i = \overline{1, m}$ (if $i = m \Rightarrow l_{i+1} = l_1$) intersect (because the sum of inner angles between these lines and B_iB_{i+1} is equal either to $2\beta < \pi$ or to $\gamma + \beta < \pi$). The point of intersection and the polygon $B_1B_2 \dots B_m$ are in different semiplanes relatively the line B_iB_{i+1} ($\beta, \gamma < \frac{\pi}{2}$). Denote $A_i = l_i \cap l_{i+1}, i = \overline{1, m}$ ($i = m \Rightarrow l_{i+1} = l_1$). Thus we see that $B_1B_2 \dots B_m$ is inscribed in a convex polygon $A_1A_2 \dots A_m$.

Now let us show that for any $m \ge 3$ there exists such α that $\angle B_2B_1B_m < \beta \Leftrightarrow$

$$\frac{\pi(m-2)}{m} - \alpha < \frac{\pi}{2} - \frac{\pi(m-2)}{2m} - \frac{\alpha}{2(m-1)} \Leftrightarrow$$

$$2\pi m - 4\pi - 2\alpha m < \pi m - \pi m + 2\pi - \alpha \frac{m}{m-1} \Leftrightarrow$$

$$\alpha \left(2m - \frac{m}{m-1}\right) > 2\pi m - 6\pi \Leftrightarrow$$

$$\alpha m \frac{2m-3}{m-1} > 2\pi(m-3) \Leftrightarrow$$

$$\alpha > \frac{2\pi(m-3)(m-1)}{m(2m-3)}$$

Notice that $\forall m \geq 3$, $\frac{2\pi(m-3)(m-1)}{m(2m-3)} < \frac{\pi(m-2)}{m}$, so $\forall m \geq 3$ any value from $\alpha \in (\frac{2\pi(m-3)(m-1)}{m(2m-3)}; \frac{\pi(m-2)}{m})$ can be selected.

Show that for every $m \geq 3$ there exists such α that $\angle B_2B_1B_m < \gamma \Leftrightarrow$

$$\frac{\pi(m-2)}{m} - \alpha < \frac{\pi}{2} - \frac{\pi(m-2)}{2m} + \frac{\alpha}{2} \Leftrightarrow$$

$$2\pi m - 4\pi - 2\alpha m < \pi m - \pi m + 2\pi + \alpha m \Leftrightarrow$$

$$\alpha > \frac{2\pi(m-3)}{3m}$$

Notice that $\forall m \geq 3$, $\frac{2\pi(m-3)(m-1)}{m(2m-3)} \geq \frac{2\pi(m-3)}{3m}$, so $\forall m \geq 3$ α can be selected from the interval $(\frac{2\pi(m-3)(m-1)}{m(2m-3)}; \frac{\pi(m-2)}{m})$.

We also show that for every $m \geq 3$ there exists such α that $\angle B_2B_1B_m < \angle B_iA_iB_{i+1}$, $i = \overline{2, m-1} \Leftrightarrow$

$$\angle B_2B_1B_m < \pi - 2\beta \Leftrightarrow$$

$$\frac{\pi(m-2)}{m} - \alpha < \pi - \pi + \frac{\pi(m-2)}{m} + \frac{\alpha}{m-1} \Leftrightarrow$$

$-\alpha < \frac{\alpha}{m-1}$. it's true $\forall \alpha > 0$.

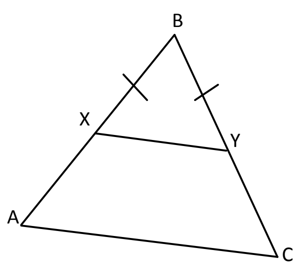
Finally, let us show that for any $m \geq 3$ there exists such α that $\angle B_2B_1B_m < \angle B_iA_iB_{i+1}$, $i = 1, m \Leftrightarrow$

$$\angle B_2B_1B_m < \pi - \beta - \gamma \Leftrightarrow$$

$$\frac{\pi(m-2)}{m} - \alpha < \pi - \frac{\pi}{2} + \frac{\pi(m-2)}{2m} + \frac{\alpha}{2(m-1)} - \frac{\pi}{2} + \frac{\pi(m-2)}{2m} - \frac{\alpha}{2} \Leftrightarrow$$

$-\alpha < \frac{\alpha}{2(m-1)} - \frac{\alpha}{2}$. it's true $\forall \alpha > 0$.

Thus we proved that for any m we can choose such value of α that $\angle B_2B_1B_m$ will be smaller than any angle of any outside triangle in $A_1A_2 \dots A_m$.



Let us call the next algorithm *the operation C with* $\angle ABC$ *of* ΔABC. Mark points X and Y on sides AB and BC respectively with the condition $XB = BY$ (lengths of XB and BY can be infinitesimal). Then we draw the segment XY and obtain angles $\angle AXY = \angle CYX = \pi - \angle BYX = \angle ABC + +\angle BXY > \angle ABC$. Take angles $\angle AXY, \angle CYX$ as a result of the operation.

Now do the operation C with $\angle B_2A_2B_3$, for example. As a result we obtain greater angles than an original. Therefore they are greater than $\angle B_2B_1B_m$. Let X and Y be new vertices of a circumscribed polygon instead o A_2. So we have a described over $B_1B_2 \dots B_m$ convex polygon with $m+1$ vertices. If we use the same operation with one of the received angles then we obtain a convex polygon witl $m+2$ vertexes. If we do so again and again then we can obtain any an arbitrary of angles of a described polygon (we can do so indefinitely many times because the lengths of XB and BY can be infinitesimal). But $\angle B_2B_1B_m$ can be smaller than any received angle due to the properties of *the operation C*. ■