Minsk, Belarus 2009

Monotonic squares

In present work we investigate the monotonic squares of natural numbers. Monotonic squares can be divided into 2 groups: increasing and decreasing. Decreasing monotonic square is such a square of a natural number, that each of its digits do not exceed the previous one (examples: 81, 8874441). In a similar manner, monotonic increasing square is such a square of a natural number that each of its digits do not exceed the following one (examples: 49, 4489, 27889).

Our task was to find all monotonic squares and to prove that no more such squares exist.

Increasing squares

With the help of a computer program (designed by ourselves) we found all the increasing squares from 1 to 10^{20}.

a^2	a	a^2	a	a^2	a	a^2	a	a^2	a
1	1	1369	37	146689	383	1111155556	33334	111333666889	333667
4	2	1444	38	344569	587	1111222225	33335	113344668889	336667
9	3	4489	67	444889	667	1111355569	33337	134444688889	366667
16	4	6889	83	2666689	1633	1113356689	33367	444444888889	666667
25	5	11236	106	2778889	1667	1133466889	33667	2777778888889	1666667
36	6	11449	107	11115556	3334	1344468889	36667	11111115555556	3333334
49	7	13456	116	11122225	3335	4444488889	66667	11111122222225	3333335
144	12	13689	117	11135569	3337	11122233444	105462	11111135555569	3333337
169	13	27889	167	11336689	3367	15666777889	125167	11111335556689	3333367
225	15	33489	183	11444689	3383	27777888889	166667	11113335666889	3333667
256	16	111556	334	13446889	3667	111111555556	333334	11133346668889	3336667
289	17	112225	335	23357889	4833	111112222225	333335	11334446688889	3366667
1156	34	113569	337	44448889	6667	111113555569	333337	13444446888889	3666667
1225	35	134689	367	277788889	16667	111133556689	333367	44444448888889	6666667

Notice that many increasing squares have the similar digital structure in decimal record and the structure of their square roots is also rather similar. Call such a group of similar numbers a *family*. We found 4 families of increasing squares, and designed formulas, which help to obtain the squares entering these families.

Here are these formulas (from now on denote the row of n digits A by $AAA\ldots(n)\ldots A$, the amount of A may vary):

$333\ldots(n)\ldots34^2 = 111\ldots(n+1)\ldots1555\ldots(n)\ldots56$

$333\ldots(n)\ldots35^2 = 111\ldots(n)\ldots1222\ldots(n+1)\ldots25$

$1666\ldots(n)\ldots67^2 = 2777\ldots(n)\ldots7888\ldots(n+1)\ldots89$

$333\ldots(n)\ldots3666\ldots(n)\ldots67^2 = 11\ldots(n-k)\ldots1333\ldots(k+1)\ldots3555\ldots(n-2k-1)\ldots$
$\ldots5666\ldots(k+1)\ldots6888\ldots(k)\ldots89,\ n > 2k$

$333\ldots(n-k)\ldots3666\ldots(n)\ldots67^2 = 111\ldots(n-k)\ldots1333\ldots(n-k)\ldots3444\ldots(2k-n+$
$1)\ldots$
$\ldots4666\ldots(n-k)\ldots6888\ldots(k)\ldots89,\ n \le 2k$

Proofs for families of increasing squares in decimal record

$333\ldots(n)\ldots34^2 = 111\ldots(n+1)\ldots1555\ldots(n)\ldots56$ (*)

Let's prove by induction that (*) is true ($n \ge 0$).

If $n = 0$ then $4^2 = 16$
If $n = 1$ then $34^2 = 1156$
Suppose (*) is true when $n = k$. Prove that (*) is true when $n = k + 1$.

$333\ldots(k+1)\ldots34 = 333\ldots(k)\ldots34 + 3*10^{k+1}$
$333\ldots(k+1)\ldots34^2 = (333\ldots(k)\ldots34 + 3*10^{k+1})^2 = 11\ldots(k+1)\ldots155\ldots(k)\ldots56 +$
$+ 2*33\ldots(k)\ldots34*3*10^{k+1} + 9*10^{2k+2}$

```
 333…(k)..34
*          6
____________
2000…(k)..04
```

```
  111…(k+1)…11555…(k)……56
+9000…….........00000…………00
+2000…(k)……04000…(k+1)…00
11111…(k+2)...15555…(k+1)…56
```

I.e. $333\ldots(k+1)\ldots34^2 = 111\ldots(k+2)\ldots1555\ldots(k+1)\ldots56$

Consequently by induction $33\ldots(n)\ldots34^2 = 111\ldots(n+1)\ldots1555\ldots(n)\ldots56$

$333\ldots(n)\ldots35^2 = 111\ldots(n)\ldots1222\ldots(n+1)\ldots25$ (*)

Let's prove by induction that (*) is true ($n \ge 0$).

If $n = 0$ then $5^2 = 25$

If $n = 1$ then $35^2 = 1225$
Suppose (*) is true when $n = k$. Prove that (*) is true when $n = k + 1$.

$333\ldots(k+1)\ldots35 = 333\ldots(k)\ldots35+3*10^{k+1}$
$333\ldots(k+1)\ldots35^2 = (333\ldots(k)\ldots35+3*10^{k+1})^2 = 11\ldots(k+1)\ldots122\ldots(k)\ldots25 +$
$+ 2*33\ldots(k)\ldots35*3*10^{k+1} + 9*10^{2k+2}$

$$
\begin{array}{r}
333\ldots(k)\ldots.35 \\
* \qquad\qquad 6 \\
\hline
2000..(k\text{-}1)..010
\end{array}
$$

$$
\begin{array}{l}
\quad 111\ldots(k)\ldots\ldots11222\ldots(k+1)\ldots25 \\
{}^{+}9000\ldots\ldots\ldots\ldots00000\ldots\ldots\ldots\ldots00 \\
{}^{+}\underline{2000\ldots(k\text{-}1)\ldots.01000\ldots(k+1)\ldots00} \\
11111\ldots(k+1)\ldots12222\ldots(k+2)\ldots25
\end{array}
$$

I.e. $333\ldots(k+1)\ldots35^2 = 111\ldots(k+1)\ldots1222\ldots(k+2)\ldots25$

Consequently by induction $33\ldots(n)\ldots35^2 = 111\ldots(n)\ldots1222\ldots(n+1)\ldots25$

<u>$1666\ldots(n)\ldots67^2 = 2777\ldots(n)\ldots7888\ldots(n+1)\ldots89$</u> (*)

Let's prove by induction that (*) is true ($n \geq 0$).

If $n = 0$ then $17^2 = 289$
If $n = 1$ then $167^2 = 27889$
Suppose (*) is true when $n = k$. Prove that (*) is true when $n = k + 1$.

$1666\ldots(k+1)\ldots67 = 1666\ldots(k)\ldots67 + 15*10^{k+1}$
$1666\ldots(k+1)\ldots67^2 = (1666\ldots(k)\ldots67 + 15*10^{k+1})^2 = 2777\ldots(k)\ldots7888\ldots(k+$
$1)\ldots89 +$
$+ 2*1666\ldots(k)\ldots67*15*10^{k+1} + 225*10^{2k+2}$

$$
\begin{array}{r}
166\ldots(k)\ldots67 \\
* \qquad\qquad 3 \\
\hline
500\ldots(k)\ldots.01
\end{array}
$$

$$
\begin{array}{l}
\quad 277\ldots(k)\ldots\ldots77888\ldots(k+1)\ldots89 \\
{}^{+}22500\ldots\ldots\ldots\ldots000000\ldots\ldots\ldots\ldots00 \\
{}^{+}\underline{\ 5000\ldots(k)\ldots\ldots01000\ldots(k+1)\ldots00} \\
277777\ldots(k+1)\ldots78888\ldots(k+2)\ldots89
\end{array}
$$

I.e. $1666\ldots(k+1)\ldots67^2 = 2777\ldots(k+1)\ldots7888\ldots(k+2)\ldots89$

Consequently by induction $1666\ldots(n)\ldots67^2 = 2777\ldots(n)\ldots7888\ldots(n+1)\ldots89$

$333\ldots(n)\ldots3666\ldots(n)\ldots672^2=111\ldots(n-k)\ldots1333\ldots(k+1)\ldots3555(n-2k-1)\ldots$
$\ldots5666\ldots(k+1)\ldots6888\ldots(k)\ldots89,\ n>2k$

$333\ldots(n-k)\ldots3666\ldots(n)\ldots672^2 = 111\ldots(n-k)\ldots1333\ldots(n-k)\ldots3444\ldots(2k-n$
$+1)\ldots4666\ldots(n-k)\ldots6888\ldots(k)\ldots89,\ n\leq 2k$

Let's prove by induction that
$333\ldots(a)\ldots3*333\ldots(b)\ldots3 = 11\ldots(a-1)\ldots1099\ldots(b-a)\ldots9888\ldots(a-1)\ldots89,\ a < b$

$333\ldots(a)\ldots3*333\ldots(b)\ldots3 = 999\ldots(b)\ldots9*111\ldots(a)\ldots1 \Rightarrow$
$333\ldots(a)\ldots3*333\ldots(b)\ldots3 = 11\ldots(a-1)\ldots1099\ldots(b-a)\ldots9888\ldots(a-1)\ldots89$ ó
$999\ldots(b)\ldots9*111\ldots(a)\ldots1 = 11\ldots(a-1)\ldots1099\ldots(b-a)\ldots9888\ldots(a-1)\ldots89$ (1)

If $a = 1$ then $999\ldots(b)\ldots9*1 = 999\ldots(b)\ldots9$
If $a = 2$ then $999\ldots(b)\ldots9*11 = 1099\ldots(b-2)\ldots989$ (**)

$$\begin{array}{rl} & 999\ldots(b)\ldots.9 \\ * & \underline{\qquad\qquad 11} \\ + & 999\ldots(b)..\ 99 \\ & \underline{999\ldots(b)\ldots\ 9\ \ } \\ & 1099..(b-2)..989 \end{array}$$

$\Rightarrow$ (**) is true

Suppose (1) is true when $a = p$. Prove that (*) is true when $a = p + 1$.

$999\ldots(b)\ldots9*111\ldots(p+1)\ldots1 =$
$999\ldots(b)\ldots9*111\ldots(p)\ldots1+999\ldots(b)\ldots9*1*10^p=$
$=111\ldots(p-1)\ldots10999\ldots(b-p)\ldots9888\ldots(p-1)\ldots89+999\ldots(b)\ldots9000\ldots(p)\ldots0$

$$\begin{array}{rl} & 111\ldots(p\text{-}1)\ldots.11099\ldots(b\text{-}p)\ldots\ldots99888\ldots(p\text{-}1)\ldots89 \\ + & \underline{999\ldots\ldots\ldots..(b\text{-}1)\ \ldots\ldots\ldots\ldots..99000\ldots(p\text{-}1)\ldots00} \\ & 1111\ldots(p)\ldots\ldots11099\ldots(b\text{-}p\text{-}1)\ldots9888\ldots..(p)\ldots..89 \end{array}$$

I.e. $999\ldots(b)\ldots9*111\ldots(p+1)\ldots1 = 11\ldots(p)\ldots1099\ldots(b-p-1)\ldots9888\ldots(p)\ldots89$

Consequently by induction
$333\ldots(a)\ldots3*333\ldots(b)\ldots3=11\ldots(a-1)\ldots1099\ldots(b-a)\ldots9888\ldots(a\text{-}1)\ldots89$

Let's prove by induction that
$333\ldots(n)\ldots37^2 = 11\ldots(n)\ldots13555\ldots(n-1)\ldots569$ (2)

If $n = 1$ then $37^2 = 1369$

Suppose (2) is true when $n = m$. Prove that (*) is true when $n = m + 1$.

$333\ldots(m + 1)\ldots37=333\ldots(m)\ldots37+3*10^{m+1}$
$333\ldots(m + 1)\ldots37^2=(333\ldots(m)\ldots37+3*10^{m+1})^2=111\ldots(m)\ldots13555\ldots(m\text{-}1)\ldots569+$
$+2*333\ldots(m)\ldots37*3*10^{m+1}+9*10^{2m+2}$

$$
\begin{array}{rr}
 & 333\ldots(m)\ldots.37 \\
* & 6 \\
\hline
 & 200\ldots(m\text{-}1)..022
\end{array}
$$

$$
\begin{array}{rr}
 & 111\ldots(m)\ldots...113555\ldots(m+1)\ldots569 \\
+ & 2000\ldots.......022000\ldots\ldots\ldots.0000 \\
+ & 9000\ldots\ldots\ldots.000000\ldots\ldots\ldots.0000 \\
\hline
 & 11111\ldots(m+1)\ldots31555\ldots(m+2)..5569
\end{array}
$$

I.e. $333\ldots(m + 1)\ldots37^2 = 111\ldots(m + 1)\ldots13555\ldots(m)\ldots569$

Consequently by induction $333\ldots(n)\ldots37^2 = 111\ldots(n)\ldots13555\ldots(n - 1)\ldots569$

$333\ldots(n)\ldots37+333\ldots(k)\ldots30 = 333\ldots(n - k)\ldots3666\ldots(k)\ldots67,\ n > k$

$A=333\ldots(n\text{-}k)\ldots3666\ldots(k)\ldots67^2=333\ldots(n)\ldots37^2+333\ldots(k)\ldots30^2+$
$+2*333\ldots(n)\ldots37*33\ldots(k)\ldots30=333\ldots(n)\ldots37^2+333\ldots(k)\ldots30^2+$
$+20*333\ldots(k)\ldots3*(333\ldots(n)\ldots30+7)= 333\ldots(n)\ldots37^2+333\ldots(k)\ldots30^2+$
$+200*333\ldots(n)\ldots3*333..(k)\ldots3+140*333\ldots(k)\ldots3=$
$=111\ldots(n)\ldots13555\ldots.(n\text{-}1)\ldots569+$
$+111\ldots(k\text{-}1)\ldots10999\ldots(n\text{-}k)\ldots9888\ldots(k\text{-}1)\ldots8900+$
$+22\ldots(k\text{-}1)\ldots21999\ldots(n\text{-}k)\ldots9777\ldots(k\text{-}1)\ldots7800(***)+4666\ldots(k\text{-}1)\ldots620(*/*)=$
$=111\ldots(n)\ldots13555\ldots(n\text{-}1)\ldots569+111\ldots(k)\ldots13555\ldots(k\text{-}1)\ldots520(**/*)+$
$+222\ldots(k\text{-}1)\ldots21999\ldots(n\text{-}k)\ldots9777\ldots(k\text{-}1)\ldots7800$

$$
\begin{array}{rr}
 & 111\ldots(k\text{-}1)\ldots10999\ldots(n\text{-}k)\ldots9888\ldots(k\text{-}1)\ldots89 \\
* & 2 \\
\hline
 & 222\ldots(k\text{-}1)\ldots21999\ldots(n\text{-}k)\ldots9777\ldots(k\text{-}1)\ldots78
\end{array}
\Rightarrow (***) \text{ is true}
$$

$$
\begin{array}{rr}
 & 333\ldots(k)\ldots33 \\
* & 14 \\
\hline
+ & 133\ldots(k\text{-}1)..\ 32 \\
 & 333\ldots(k)\ldots..33 \\
\hline
 & 4666\ldots(k\text{-}1)..62
\end{array}
\Rightarrow (*/*) \text{ is true}
$$

111…(k-1)…10888…(k-1)…8900
+ 466..…(k-1)...6620
111…(k)……11355…(k-1)…5520 => (**/*) is true

Consider 3 variants: $n > 2k$, $n = 2k$ and $n < 2k$

1. $n > 2k$

222…(k-1)…221999999999999…(n-k)…99777…(k-1)…77800
+ 1…(k)…....11355…(k-1)…55520
222…(k)…...2220…(n-2k)....01…(k+1)…11133…(k-1)…33320

1111…….11…(n)……1355555555555555…(n-1)….55...……….569
+ 2…(k)……200…(n-2k)… …01…(k+1)…13…(k-1)…320
1…(n-k)….3…(k+1)…335…(n-2k-1)….56….(k+1)...68…(k)……889

=> if $n>2k$ then
333…(n-k)…3666…(k)…67^2=111…(n-k)…1333…(k+1)…3555…(n-2k-1)…
…5666…(k+1)…6888…(k)…89

2. $n = 2k$

222…(k-1)…2219…(n-k)…99777…(k-1)…77800
+ 1…(k)…....11355…(k-1)…55520
222…(k)…...2221…(k+1)…11133…(k-1)…33320

1111…….11…(n)……135…(n-1)….55...……..569
+ 2…(k)……211…(k+1)…13…(k-1)…320
11….(k)….3…(k)…..3346…(k)……68….(k)......889

=> if $n > 2k$ then
333…(n-k)…3666…(k)…67^2=111…(k)…1333…(k)…34666…(k)…6888…(k)…
…89

3. $n < 2k$

222…(k-1)…22………22199….(n-k)…9777…(k-1)…7800
+ 1...(k)…11111…............ 1355…(k-1)….5520
2…(n-k)...…23..(2k-n)..3311..(n-k+1)..1133…(k-1)….3320

111111111111……...(n)…………….11135……...(n-1)…………569
+ 2 ...(n-k+1).23…(2k-n)..…..311..(n-k+1).13…(k-1).....320
1…(n-k)….13…(n-k)…34…(2k-n+1)….446….(n-k)...68…(k)……889

=> if $n < 2k$ then
333…(n-k)…3666…(k)…67^2=111…(n-k)…1333…(n-k)…3444…(2k-n+1)…

…4666…(*n*-*k*)…6888…(*k*)…89

But except the increasing squares which belong to the families described above, there are some numbers which belong to none of the families. We believe that the quantity of such squares if finite, because there are only 22 such squares among the numbers from 1 to 10^{20}, and the greatest of them is eleven-value.

Here are the increasing squares which are not entering into families:

1,4,9,36,144,169,225,256,1444,6889,11236,11449,13456,13689,33489,146689, 344569,2666689,11444689,23357889,11122233444,15666777889.

Decreasing squares

With the help of a computer program we found all the decreasing squares from 1 to 10^{20}.

1	44100	64000000	10000000000	8410000000000
4	84100	77440000	40000000000	8874441000000
9	90000	81000000	44100000000	9000000000000
64	96100	100000000	84100000000	9610000000000
81	640000	400000000	88744410000	9853321000000
400	774400	441000000	90000000000	9998876410000
441	810000	841000000	96100000000	64000000000000
841	1000000	887444100	98533210000	77440000000000
900	4000000	900000000	99988764100	81000000000000
961	4410000	961000000	640000000000	100000000000000
6400	8410000	985332100	774400000000	400000000000000
7744	8874441	999887641	810000000000	441000000000000
8100	9000000	6400000000	1000000000000	841000000000000
10000	9610000	7744000000	4000000000000	887444100000000
40000	9853321	8100000000	4410000000000	900000000000000

It’s easy to see that the squares which end with zero can be obtained from the ones which do not end with zero by simply multiplicating the square root of the number by 10 to some natural power. Therefore the number of increasing squares is infinite, but we can construct all the decreasing squares from the ones which do not end on zero.

From now on we can consider only those decreasing squares which last digit isn’t zero. Thus, there remains only 12 decreasing squares among the numbers from 1 to 10^{20} and the greatest of them is nine-value:

1,4,9,64,81,441,841,961,7744,8874441,9853321,999887641

Other notations

In various notations (from binary to nonary) proofs for the families of increasing squares, for the increasing squares which are not entering into families and for the decreasing squares are rather similar to the ones in decimal notation.

It has been investigated that for all the notations (from binary to decimal) there only a few of decreasing squares not ending with zero. In three notations (quinary, nonary and decimal) some families of increasing squares were obtained. In other notations the number of increasing squares does not exceed 12.

Notation	Number of decreasing squares without zero	Number of families	Number of the increasing squares which are not entering into families
Binary	1	None	1
Ternary	4	None	3
Quaternary	3	None	1
Quinary	6	1	7
Sextuple	6	None	10
Septenary	10	None	12
Octuple	19	None	9
Nonary	9	2	13
Decimal	10	4	22

Results of research in other notations

Binary
Decreasing:
$1*10^n$
Increasing:
1

Ternary
Decreasing:
$1*10^n$, $11*10^n$, $221*10^n$, $11111*10^n$,
Increasing:

1, 11, 11111

Quaternary
Decreasing:
$1*10^n$, $21*10^n$, $2221*10^n$,
Increasing:
1

Quinary
Decreasing:
$1*10^n$, $4*10^n$, $31*10^n$, $311*10^n$, $441*10^n$, $33111111*10^n$,
Increasing:
1, 4, 144, 224, 2244, 11114, 22234444, 11…(n + 1)…133…(n)…34=2…(n)…23^2

Sextuple
Decreasing:
$1*10^n$, $4*10^n$, $41*10^n$, $321*10^n$, $441*10^n$, $5555321*10^n$,
Increasing:
1, 4, 13, 24, 144, 244, 3344, 11224, 13444, 112244

Septenary
Decreasing:
$1*10^n$, $4*10^n$, $22*10^n$, $51*10^n$, $331*10^n$, $441*10^n$, $642*10^n$, $42222*10^n$, $55411*10^n$, $5533221*10^n$,
Increasing:
1, 4, 12, 22, 34, 144, 1111, 4444, 11122, 11334, 233344, 111144444444

Octuple
Decreasing:
$1*10^n$, $4*10^n$, $11*10^n$, $31*10^n$, $44*10^n$, $61*10^n$, $441*10^n$, $551*10^n$, $744*10^n$, $3221*10^n$, $5544*10^n$, $6444*10^n$, $6631*10^n$, $7211*10^n$, $54421*10^n$, $77771*10^n$, $444411*10^n$, $666644*10^n$, $5544444*10^n$,
Increasing:
1, 4, 11, 44, 144, 1244, 3344, 11444, 114444

Nonary
Decreasing:
$1*10^n$, $4*10^n$, $54*10^n$, $71*10^n$, $441*10^n$, $764*10^n$, $831*10^n$, $5551*10^n$, $65311*10^n$,
Increasing:
1, 4, 17, 144, 1134, 1277, 1357, 3377, 11127, 13337, 1122257, 6666677, 33347777,
222…(n+1)…266…(n)…67=44…(n)…45^2
222…(n+1)…2366…(n)…67=144…(n)…45^2