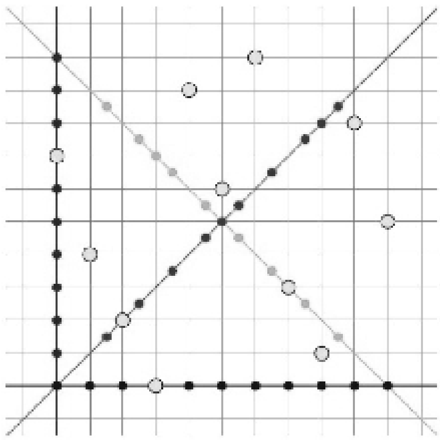
Minsk, Belarus 2009

Problem 10

Centro-Symmetric Shadows

A set of points in the space is called *centro-symmetric* if it has a centre of symmetry. (Recall that a *centre of symmetry* of a set *S* in the space is a point *C* with the property that for any point $P \in S$ there exist a point $P' \in S$ such that *P* and *P'* are equidistant from *C* and lie on a line passing thought *C*).

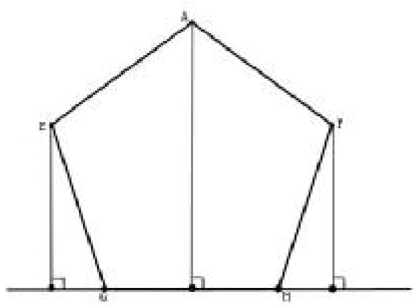


Firstly consider case when the set of points is in the plane. Given a set of points, its *shadow* on a line is its orthogonal projection onto this line (similarly *shadow of the set in space* is its orthogonal projection onto some plane).

Consider a positive integer $n \geq 3$. Denote by *k(n)* the minimum positive integer *k* with the following property: for any set *S* of *n* points in the plain, if there exist *k* lines, no two parallel, such that for each line the shadow of *S* on this line is centro-symmetric, then the initial set *S* is also centro-symmetric. Let's find the number *k(n)*.

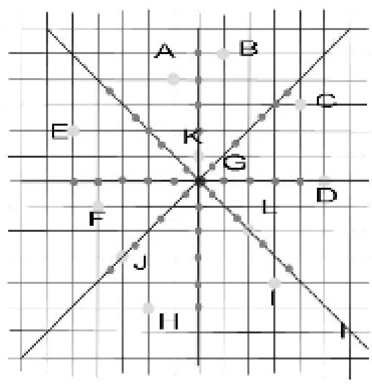
Divide the problem into two cases:

a) Firstly let *n* be odd. Then let's estimate *k(n)* in this case. Firstly show that $k(n)^{3}n+1$. Consider a set *S'* of *n* points which are the vertices of an equilateral polygon.



Since *n* is odd the set *S'* is not centro-symmetric. However the shadow of the set *S'* is centro-symmetric if the line is parallel to one of the edges of the polygon. So there exist at least *n* lines such that the shadows of *S'* on these lines are centro-symmetric. Thus $k(n)^{3}n+1$.

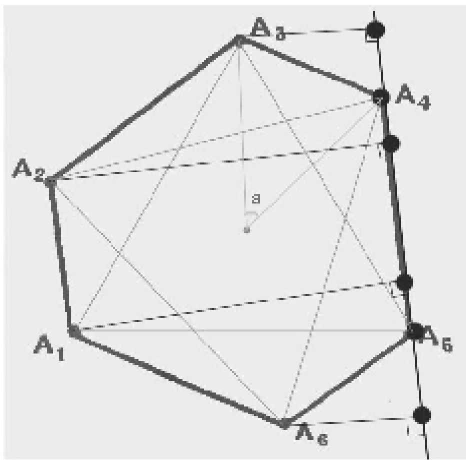
Now let's show that $k(n) \leq n+1$. Note that the projection of the gravity centre onto some line (denote gravity centre of *S* by *G*) is the gravity centre of this projection. Hence we can move lines parallelly to themselves. Move all the lines in order to make the gravity centers of projections and the point *G* coincident. Therefore without loss of generality assume that all *k* lines are crossing into the one point *G*.



If the set *S* is centro-symmetric then points *G* and *C* coincide (denote by *C* symmetry center of of *S*). Suppose there is a pair of points such that their projections onto at least two lines are symmetric. Then this pair of points is symmetric with respect to *G*.

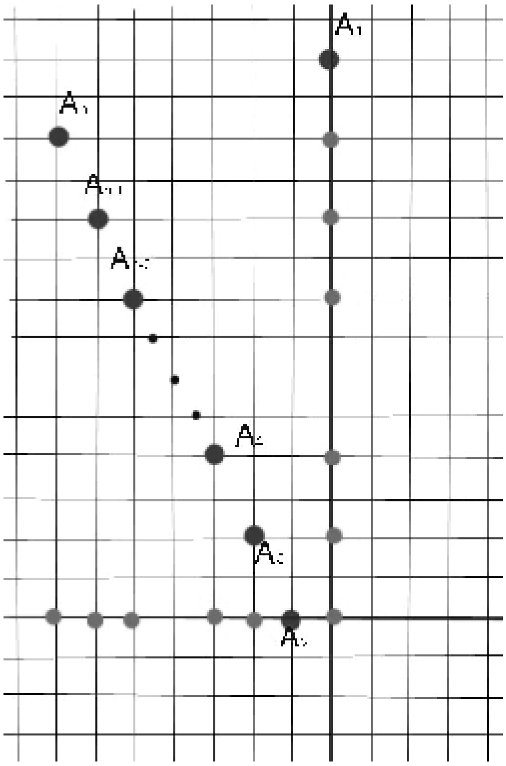
If there exists no such pair of points then all points of the set will break into pairs which projections are symmetric (for example on the figure you can see the projections of points *E* and *I*, *A* and *D*, *L* and *B*, *F* and *H*, *C* and *K* onto line *l* are symmetric and the projection of point *J* onto the same line and the point *G* coincide). Let's find how many different partitions into such pairs exist. Every point can form a pair with *(n-1)* other points or be without a pair (as *n* is odd). So there are *n* different partitions into pairs such that

no pair repeats. Then there are at most *n* lines, such that for each line the shadow of the set on this line is centro-symmetric. Suppose there are *(n*+1*)* such lines. Then there is at least one pair of points such that these points are symmetric with respect to *G*. Therefore their projections are symmetric with respect to each line and we can exclude these points and all their projection out of consideration. Hence we have *(n*-2*)* points left in the set. If there are no pairs of points which are symmetric with respect to *G* then there will be at most *(n*-2*)* lines, such that for each line the shadow of the set is centro-symmetric. But we still have *(n*+1*)* such lines. It means that there is at least one pair of points such that this points are symmetric with respect to a point *G* and we can exclude it from the set as above. There are *(n*-4*)* points left. Continuing in the same way, we see that whole set is divided into the pairs of the points which are symmetric with respect to a point *G*. In the other words the set *S* is centro-symmetric. So we've proved that *k(n)*£*n*+1. Finally we have *k(n)*³*n*+1 and *k(n)*£*n*+1. So *k(n)*=*n*+1 for any odd *n*.
b) Now let *n* be even. Estimate *k(n)* in this case. Like in the case of odd *n* suppose we have a set of points such that there is no pair of points which are symmetric with respect to a point *G*. Every point of the set can form a pair with *n-1* other points and can't be without a pair (since *n* is even). Then the same way as in the item a) we have that there are not more than *n-1* lines, such that for each line the shadow of the set is centro-symmetric. Thus we obtain that *k(n)*£*n*.
We can represent every natural number *n* as $n=2^l*(2*m+1)$ (where *l, m* are integer and *l, m*≥0). Then show that *k(n)*³$n/2^l+1$. Construct a set of points *S"* which isn't centro-symmetric and has $n/2^l$ lines, such that for each line the shadow of the set is centro-symmetric. Consider equilateral polygon with *(*2*m*+1*)* vertices. Then rotate it with respect to its middle point on a little angle 2^l-1 times (we will call such a figure a semi-equilateral polygon).



Let's prove that vertices of such a semi right polygon are not a sentro-symmetric set. Let's take one of the *n* vertices of the semi right polygon (denote it by A_i). If the set which contain its vertices is sentro-symmetric it will be one more point from this set (denote it by A_j) which lie on a line passing through the middle point of this polygon (denote it by M) and point A_j. The point A_i belongs to one of the right polygon with *(2*m+1)* vertices which middle point and middle point M of the semi right polygon are coincide. Let's draw a line which is passing through points M and A_i. It will cross the opposite side of the right polygon with *(2*m+1)* vertices as it has odd quantity of vertices. The angle between this line and the line passing through M and one of the points bordering on this opposite side is equal to *360⁰/(4*m+2)*. Thus to have the point A_j we have to turn the right polygon with *(2*m+1)* vertices on an angle *360⁰/(4*m+2)*. Then we can choose such a little angle that the set of points won't be centro-symmetric. The statement is proved.
On the figure you can see the semi right polygon with 6 vertices. But projections of this vertices onto the *(2*m+1)* lines passing in parallel to the biggest edges of the semi right polygon are centro-symmetric. Let us prove this statement. Let's denote the angle with vertice at the point M and edges passing through the points which border on one edge of the semi right polygon by "midangle" of this edge. Than we can note that all "midangles" of the biggest edges of the semi right polygon are equal and all "midangles" of the other edges of the semi right polygon are equal too (it follows from construction of this set of points). Then draw the line passing through M perpendicular to the one of the biggest edges A_iA_{i+1} of the semi right polygon (denote it by *l*). As from construction all the segments A_jM are equal (A_j any vertice of this polygon) and triangle

$A_iA_{i+1}M$ is isosceles. Then line *l* is median and bisector of triangle $A_iA_{i+1}M$ so on the line *s* which the edge A_iA_{i+1} belongs to points A_i and A_{i+1} are symmetric concerning a point of crossing lines *l* and *s*. As line *l* is the bisector of the angle A_iMA_{i+1} and "midangles" of the edges $A_{i+1}A_{i+2}$ and $A_{i-1}A_i$ is equal we get l is bisector of the angle $A_{i+2}MA_{i-1}$. And as $A_{i+2}M= MA_{i-1}$ it follows that projections of points A_{i+2} and A_{i-1} on the line *s* are symmetric concerning a point of crossing lines *l* and *s*. Continuing in the same way, we see that all the set break into the pairs of the points projections of which onto the line *s* is symmetric concerning a point of crossing lines *l* and *s*. The statement is proved. Thus we have $k(n)^3 n/2^l+1$.



It's necessary to note that for any *n* $k(n)\geq 3$. Let's construct the set of n points which has symmetric shadows on 2 lines to show it. You can see such a set on this figure. Points $A_2, A_3, \ldots, A_{n-2}, A_{n-1}, A_n$ lie on one line so that $A_2A_3= A_4A_5=A_6A_7= A_8A_9=\ldots=A_{n-3}A_{n-2}= A_{n-1}A_n$. Thus projections of this points onto some line will divide this line into the equal segments. Then we can choose point A_1 so that the shadows of this set of points on two perpendicular lines will be centro-symmetric (as on this figure).

Main results:

In our report we have considered sets of points in the plain and estimated number *k(n)* with the property that for any set *S* of n points in the plain, if there exist *k* lines, no two parallel, such that for each line the shadow of *S* on this line is centro-symmetric, then the initial set *S* is also centro-symmetric.

- We find $k(n)=n+1$ for any odd *n* and
- $n/2^l+1£k(n)£n$ for any even *n*.
- for every n we show $k(n)\geq 3$
- For set S of n points in the space we proved $k(n)\leq n+1$ for any odd *n* and $k(n)£n$ for any even *n*.