Minsk, Belarus

Problem 1.
Specular Colorings

Contents:

The statement of the problem

1. What is the maximum number of cells of an m× n grid that can be coloured blue, such that no two blue cells are symmetric with respect to any horizontal or vertical line of the grid?

2. Some cells of an $m \times n$ grid are coloured blue. We call such a colouring specular if for any interior horizontal or vertical line of the grid there are two blue cells that are symmetric with respect to this line. Denote by $S(m, n)$ the minimal number of blue cells in a specular colouring of an $m \times n$ grid.

Find $S(m, n)$ or estimate it (give lower and upper bounds).

3. Formulate and investigate 3-dimensional analogs of the problem.

2. Point 1.

Consider this problem for n-dimensional case. An n-dimensional table is given. It is required to color the maximum number of cells in the n-dimensional table so that for any (n-1)-dimensional hyperplane which is passing between cells of the table, there is no 2 symmetrically colored cells. We say that cells $(b_1, b_2, ..., b_n)$ and $(c_1, c_2, ..., c_n)$ are symmetric with respect to (n-1)-dimensional hyperplane $x_i = k$, where x_i - i-th coordinate of the n-dimensional space, when $\forall j \neq i, b_j = c_j; c_i = 2k - b_i$.

Prove that for the n-dimensional table $a_1 \times a_2 \times ... \times a_n$, where $a_i > 1, 1 \leq i \leq n$, the maximum is $\left[\frac{a_1 a_2 ... a_n + 1}{2}\right]$.

Consider the case n=1: if more than $\left[\frac{a_1+1}{2}\right]$ cells are colored, then there exist two colored cells such that they are symmetrical with respect to a line passing through some border between them. We can always color $\left[\frac{a_1+1}{2}\right]$ cells with chess coloring, so for n=1 $\left[\frac{a_1+1}{2}\right]$ - maximum.

Suppose the formula is proved for n=k, then prove it for n=k+1. Consider a_{k+1} k-dimensional tables $a_1 \times a_2 \times ... \times a_k$. Assume that T cells are colored in the first table, then in the second one no more than $a_1a_2...a_k - T$ cells can be colored (otherwise in the first and second table there will be two symmetrically painted cells), in third one - not more than $a_1a_2...a_{n-1} - (a_1a_2...a_k - T) = T$, etc.

1) a_{k+1} – even. There can be colored not more than

$$\frac{a_{k+1}\ +1}{2}(T + (a_1a_2...a_k - T)) = \frac{a_1a_2...a_ka_{k+1}}{2} = \left[\frac{a_1a_2...a_ka_{k+1}+1}{2}\right]$$

cells in the whole table.

2) a_{k+1n} – odd. There can be colored no more than

$$\frac{a_{k+1}-1}{2}(T + (a_1a_2...a_k - T)) + T = \frac{a_1a_2...a_ka_{k+1} - a_1a_2...a_k}{2} + T \le \frac{a_1a_2...a_ka_{k+1} - a_1a_2...a_k}{2} + \left[\frac{a_1a_2...a_k\ +1}{2}\right] =$$

$$\left[\frac{a_1a_2...a_ka_{k+1} - a_1a_2...a_k + a_1a_2...a_k + 1}{2}\right] = \left[\frac{a_1a_2...ak_{k+1}+1}{2}\right]$$

cells in the whole table, because $T \le \left[\frac{a_1a_2...a_k+1}{2}\right]$.

In both cases we can reach the estimation using chess coloring. If the sum of coordinates of cell is even than we color this cell (coordinates start from 0).

Point 2.

Divide the problem on two subtasks:

2.1. Consider the given problem for a string of n cells.

2.2. Prove that for the 2-dimensional table the minimum is reached, if we take the first column and line and we color them using the results of item 2.1.

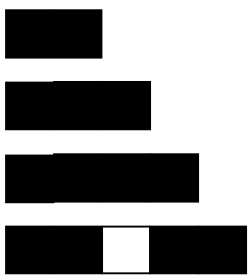
Then we obtain that the minimum for the table is S(m, n)=g (n) +g (m)-1, because the cell (1,1) is colored both in the column and in the line.

Point 2.1.

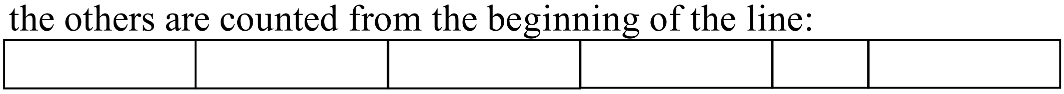
A string of n cells is given. It is required to color the minimum number of cells so that for every border between cells there is at least one symmetric pair of coloured cells

Denote the minimum by g(n).

Algorithm of colouring:



For $n > 5$ take the maximal even k such that $n \ge k^2$. If $n = k^2 + 3k + 4$ or $n = k^2 + 3k + 4$ then take k+2. Split the line into groups of k cells, where one group consists of last k cells, and the others are counted from the beginning of the line:



Color all the odd cells in the first and last groups, all cells of a kind mk+2, except last group, and the last even cage in the last group. If $n \neq rk+2$ and $n \neq tk+3$, then we also color the first even cell in the last group.

1) g(2)=2; g(3)=3; g(4)=4; g(5)=4.

2) n>5.

n	$g(n)$
k^2	$2k+1$
k^2+1	$2k+1$
k^2+2	$2k+1$
k^2+3	$2k+1$
k^2+4	$2k+2$
k^2+5	$2k+2$
…	…
k^2+k+2	$2k+2$
k^2+k+3	$2k+2$
k^2+k+4	$2k+3$
k^2+k+5	$2k+3$
…	…
k^2+2k+2	$2k+3$
k^2+2k+3	$2k+3$
k^2+2k+4	$2k+4$
k^2+2k+5	$2k+4$
…	…
k^2+3k+2	$2k+4$
k^2+3k+3	$2k+4$
k^2+3k+4	$2k+4$
k^2+3k+5	$2k+4$
k^2+3k+6	$2k+5$
k^2+3k+7	$2k+5$
…	…
k^2+4k+2	$2k+5$
k^2+4k+3	$2k+5$

For the even $k \geq 2$.

Let's analyse the table.

For $n \geq 4$:

Firstly we have 2 numbers with g(n) =4, then 2 numbers with g(n) =5, then 2 numbers with g(n) =6, then 2 numbers with g(n) =7, then 4 numbers with g(n) =8, etc:

range with one value	k^2+4 … k^2+k+3	k^2+k+4 … k^2+2k+3	k^2+2k+4 … k^2+3k+5	k^2+3k+6 … k^2+4k+3	$(k+2)^2$ … $(k+2)^2+3$
length of	k	k	k+2	k-2	4

range				k+2

Thus we have:

g(n)	4	5	6	7	8	9	10	11	12	13	14	15	
count of n	2	2	2	2	4	4	4	4	6	6	6	6	

		column

Suppose n is in t group. Then $8\frac{t(t-1)}{2} \le n-4$. Hence $t = \left[\frac{\sqrt{n-3}+1}{2}\right]$. Now suppose n is in l column (columns can be 0,1,2,3) then $8\frac{t(t-1)}{2} + 2tl \le n-4$, hence $l = \left[\frac{n-4-4t(t-1)}{2t}\right]$. Therefore g (n) =4t+l.

So the final formula looks as follows:

$$g(n) = 4\cdot\left[\frac{\sqrt{n-3}+1}{2}\right] + \left[\frac{n-4\cdot\left[\frac{\sqrt{n-3}+1}{2}\right]\cdot\left[\frac{\sqrt{n-3}-1}{2}\right]-4}{2\cdot\left[\frac{\sqrt{n-3}+1}{2}\right]}\right].$$

Prove that we cannot color a line for g (n)-2. If a even and b odd cells are colored, then $ab \le \left(\frac{a+b}{2}\right)^2$. So if we have the fixed a+b then the best choice is to color the equal number even and odd cells. Coloring 2k cells we can color the line with not more than k^2 cells, coloring 2k+1 cells - not more than k^2+k cells, coloring 2k+2 cells - not more than k^2+2k+1 cells, coloring 2k+3 cells - not more than k^2+3k+2 cells. For 2k we can color a line with not more than k^2 cells, but k^2 is even, and when n is even, pairs (1, n) and (2, n-1) satisfy one border, what means that for 2k we can color not more than k^2-1 line, for 2k+1 - not more than k^2+k-1 line, for 2k+2 - not more than k^2+2k+1 line, for 2k+3 - not more than k^2+3k+1 line. Thus we proved that the following numbers cannot be coloured with g(n)-1 cells colored:

$k^2, k^2+1, k^2+2, k^2+3, k^2+k, k^2+k+1, k^2+k+2, k^2+k+3, k^2+2k+2, k^2+2k+3$, $k^2+3k+2, k^2+3k+3, k^2+3k+4, k^2+3k+5$. The other ones cannot be coloured with g(n)-2.

Point 2.2.

2.2. Let's prove that for table $m \times n$ we must place blue squares in the one column and the one row of this table. Firstly, call vertical or horizontal line**good** if there exist two blue squares symmetrical to it. Then, let's denote:

$f_n(x)$ is the maximum number of vertical (horizontal) lines we can make good if we place x blue cells in one row (column) 1xn.

$g(n)$ is the minimum number of blue cells to make good all the verticallines in table 1xn..

So we must prove that $g(m) + g(n) - 1$ is the minimum number of blue cages that are needed to make all the lines in the table m x n good.

It's easy to prove that we can make good table m x n with $g(m) + g(n) - 1$ blue squares. Place $g(m)$ blue cells in the first row and g(n) blue cells in the first column. Then all the lines are good and we use $g(m) + g(n) - 1$ blue squares.

To prove that this number is minimum we place blue cells in the first column and the first row as we said before. Suppose there are a_1 blue cells in the first column, a_2 in the second, …, a_m in the m-th (so that first $a_1 = g(n)$ and a_i=0 or a_i=1, if $2 \leq i \leq m$). Let's change the disposition of blue cells. If we can make all lines good and use less than $g(m) + g(n) - 1$ blue cells, then we can make some disposition when we can cut one blue cell, and all lines will be good as before.

First, if all lines are good, then:

Property 1. $f_n(a_1) + f_n(a_2) + \ldots + f_n(a_m) \geq n-1$.

Property 2. There exist $g(m)$ columns, such that in any of them there exist one blue cell.

Let's prove that we can't get some disposition and cut some blue cells with both of properties realized. Make motions such way: on any motion we take one blue cell from the first column, cut it and paste it in another column (assume that we can change disposition of blue cells in any column as we want). We don't change the disposition of blue cells in the first row (because we can't do it realizing both of the properties). Suppose the first column contains not less blue cells than in any other column after every motion. It's easy to prove that if $k<g(n)-2$ then

1) if $k = 4t$ then $f_n(k) = k^2/4$;

2) if $k = 2t + 1$ then $f_n(k) = (k^2-1)/4$

3) if $k = 4t+2$ then $f_n(k) = k^2/4$ or $k^2/4-1$.

Prove it for $k = 4t$. Firstly $f_n(k) \leq k^2/4$. If we have x blue squares on the even positions and y – on odd positions, then the maximum number of lines we can make good is xy, and $xy \leq \quad \leq k^2/4$. We can make $k^2/4$ squares good if we place blue cells in cells with numbers 2, 4, 6, …,$2t$, $n - 1$, $n - 3$, …, $n - 2t + 1$ for odd n and 2, 4, 6, …,$2t$, n, $n - 2$, …, $n - 2t$ for even n and in cells with number 1, $1 + 2t$, $1 + 4t$, …, $1 + 4t^2 - 2t$. If $k \neq 4t$ then proof is similar.

Thus we can say that if x > y + 1 then $f(x) - f(x - 1) \geq f(y) - f(y - 1)$. It's easy to see that the sum $f(a_1) + +f_n(a_2) + \ldots + f_n(a_m)$ decreases on any motion after the third motion (because $f(x) - f(x - \quad -1) \geq f(y) - f(y - 1)$ is true if $x < g(n) -2$). It's also easy to see that after the first motion $f_n(a_1) + \quad + f_n(a_2) + \ldots + f_n(a_m) \leq n-1$ and we can't cut any blue cell (because $f_n(a_1) \leq n - 1$ and $f_n(a_2) + \ldots + f_n(a_m) \leq 1$). So we also have to prove that we can't cut any blue cells after the second and after the third motion.

To prove it let's first make first three motions and then make the third and the second motions back. Let's prove that the sum $f(a_1) + f_n(a_2) + \ldots + f_n(a_m)$ isn't decreasing when we make third and second motions back. It's easy to see that $f_n(a_2) + f_n(a_3) + \ldots + f_n(a_m)$ after the first motion is less than 2, after the second less then three, after the third less then five. So we must prove that $f(a_1)$ increases faster than $f_n(a_2) + f_n(a_3) + \ldots + f_n(a_m)$ decreases.

Let's prove for the case if n is odd and $g(n) = 4t + 4$ (for other cases the proof will be similar). Let before we took back the third and the second motions we made blue cells number 2, 4, 6, …,$2t$, $n - 1$, $n - 3$, …, $n - 2t + 1$ and 1, $1 + 2t$, $1 + 4t$, …, $1 + 4t^2 - 2t$. When we are making motions back we'll make blue cells number $1 + 4t^2 - t$ and number $1 + 4t^2$. Thus we get what was required.

The *n*-dimensional case

Consider n-dimensional case of the problem. Prove that for a_1 x a_2 x…x a_n the right disposition is the same as in 2-dimensional case and the minimum number of blue cells is $g(a_1) + g(a_2) +\ldots+ g(a_n) - (n - 1)$. Prove it by induction. The basis is item 2.2. Suppose that for some $n = k$ we proved that minimum number of blue cells we can use with k-dimensional grid a_1 x a_2 x…x a_k is $g(a_1) + g(a_2) +\ldots+ g(a_k) - (k - 1)$. Then let's prove the same property for $k + 1$-dimensional case. Suppose we have a grid a_1 x a_2 x…x a_{k+1}. Consider a_2 x a_3 x …x a_{k+1} columns of length a_1. First we colors $g(a_1) + g(a_2) +\ldots+ g(a_{k+1}) - k$ as we said before, so first in one column is $g(a_1)$ blue cells and in any other column 0 or 1 blue cells. It's easy to see that there exist $g(a_1) + g(a_2) +\ldots+ g(a_k) - (k - 1)$ columns such that in any of them there is one or more blue cells (we proved it before). So we can fix a disposition of all blue cells except $g(a_1)$ blue cells in one column. Let's do the same motions as we said before. If we can solve the problem and use less blue cells that we use now then after some motions we can cut one of blue cells. Let's enumerate the columns with numbers from 1 to $a_2a_3\ldots a_{k+1}$. Then if after some motions we can cut one of blue cells then after it $f_{a1}(a_2) + f_{a1}(a_3) +\ldots+ f_{a1}(a_{a2a3\ldots ak+1}) \geq a_1 -- 1$. But in item 2.2. we proved it's impossible, so we can't cut any blue cell. So we proved that for grid a_1x a_2 x…x a_{k+1} we have to use $g(a_1) + g(a_2) +\ldots+ g(a_{k+1}) - (k)$ blue cells.