10^{TH} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

QUIZ

- Each team (high school students only) is gathered in a separate room and works together. Written materials, electronics, literature or other sources are forbidden during the quiz, as well as any external help. Only brochures of the ITYM and paper language dictionaries are allowed.
- Different problems must be solved in **different** sheets of paper.
- Indicate the **problem number** and page numbers on every solution.
- Please **don't** mention your country, team or other names anywhere.

Good luck!

Problem 1. Maximal Cardinalities of Bounded-angled Sets

1. Let $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ be vectors in $\mathbb{R}^d$.

(a) Define *the Euclidean distance* between x and y. What is the greatest possible distance between them if all the entries x_i, y_i are either 0 or 1? **(1 point)**

(b) Define *the scalar product* and *the angle* between x and y. **(1 point)**

2. Fix $d \geq 2$.

(a) Determine the smallest value of $\alpha > 0$ for which $f_\alpha(d) < g_\alpha(d)$. **(1 point)**

(b) Let $X \subset \mathbb{R}^d$ denote the vertex set of a $(d-1)$-dimensional hypercube in $\mathbb{R}^d$, let $x \in X$ be an arbitrary vertex and fix $\varepsilon > 0$. Prove that there exists $x' \in \mathbb{R}^d$ such that the distance between x and x' is at most ε and the angles $\angle x'yz$, $\angle yx'z$ are strictly less than $\frac{\pi}{2}$ for all distinct $y, z \in X \setminus \{x\}$. **(3 points)**

3. Given $d \geq 2$ and $\alpha \in (\arccos(\frac{1}{4}), \frac{\pi}{2})$, let $m < \frac{2^d}{3}$ be a positive integer such that

$$3\binom{3m}{3}\left(\left(\frac{1}{4\cos\alpha}\right)^{\cos\alpha}\left(\frac{3}{4(1-\cos\alpha)}\right)^{1-\cos\alpha}\right)^d < m$$

Show that there is a set of points in $\mathbb{R}^d$ of cardinality $2m$ in which any angle formed using three of the points is strictly less than α. (Hint: consider the d-dimensional unit hypercube $C = \{0,1\}^d$.) **(4 points)**

Remark. You may use without proof the following result. If $X_1, X_2, \ldots, X_d$ are independent and identically distributed random variables attaining the values 0 and 1 only, then

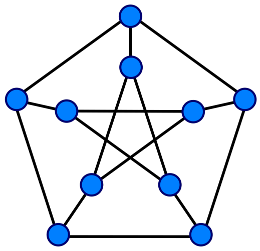
$$\mathbb{P}\left(\frac{1}{d}\sum_{i=1}^{d} X_i \leq p - \varepsilon\right) \leq \left(\left(\frac{p}{p-\varepsilon}\right)^{p-\varepsilon}\left(\frac{1-p}{1-p+\varepsilon}\right)^{1-p+\varepsilon}\right)^d$$

where $p = \mathbb{P}[X_i = 1]$ and $\varepsilon > 0$.

Problem 2. Coloring Graphs

1.

(a) What is the maximal independent set of the Petersen graph: **(1 point)**



(b) What is the chromatic number of this graph? **(1 point)**

2. Compute (with the proof) the value of $F_v(4, 4, 20)$. **(3 points)**

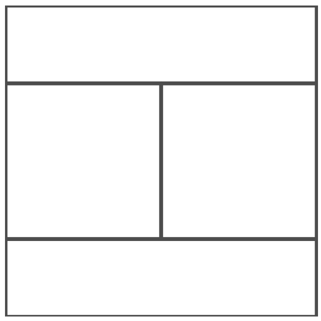
3. 66 mathematicians attend a conference. It is known that each two of them talk to each other in exactly one of four different languages. (The same person may talk different languages to others, but any two mathematicians always speak the same language to each other). Prove that there exist four mathematicians who talk to each other in the same language. **(5 points)**

Problem 3. Cutting a Rectangle

1.

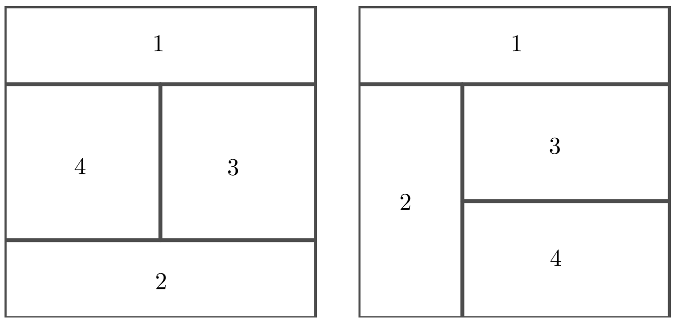
(a) Let G be a group acting on a set M via the action $\varphi : G \times M \to M$. What is the orbit of $m \in M$ under φ? **(1 point)**

(b) Let R be a rectangle with center O and denote by R_α the rotation of α degrees with center O. Define G to be the group of maps $\{R_0, R_{90}, R_{180}, R_{270}\}$ under composition. Set $M = S_{R,4}$ and $m \in M$ be the cut shown bellow. Consider the action $\varphi : G \times M \to M$ given by $\varphi(R_\alpha, m) = R_\alpha(m)$. Draw the elements of the orbit of m under the action φ. **(1 point)**



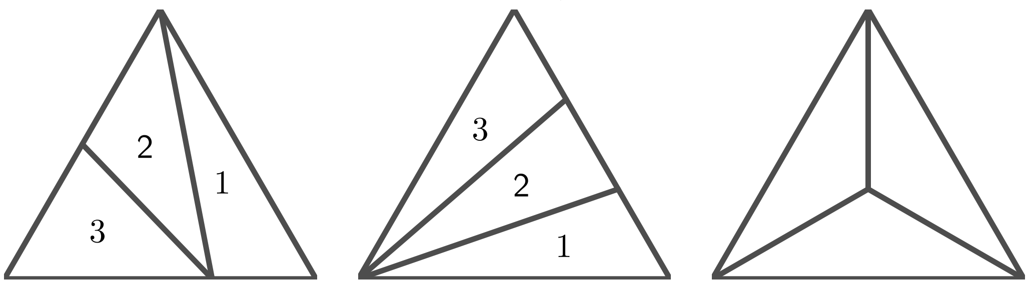
2. Let $L_{R,n} \subset S_{R,n}$ be the set of cuts from $S_{R,n}$, which can be constructed without a cut from the left. More formally, $L_{R,n} = \{x \in S_{R,n} \mid \exists$ ordering of the elements in x : $r_1, r_2, ...r_n$ such that $\forall i, \bigcup_{j=i}^{n} r_j$ is a rectangle and no $i < n$ is such that r_i contains the whole left side of $\bigcup_{j=i}^{n} r_j$ $\}$.

Provide a formula for $|L_{R,n}|$. **(3 points)**



For example, the cut on the left is from $L_{R,4}$. But the cut on the right is not from $L_{R,4}$, because of cut 2 which is from the left. Formally, for all orderings r_1, r_2, r_3, r_4, which satisfy the condition to be in $S_{R,4}$, r_2 contains the whole left side of $\bigcup_{j=2}^{4} r_j$.

3. Let T be a triangle with area 1. Let $S_{T,n}$ be the set of cuts of a triangle into n triangles of equal area, which can be constructed by cutting of a triangle of area $\frac{1}{n}$ with a segment from vertex to opposing side. Provide a formula for $|S_{T,n}|$. **(5 points)**

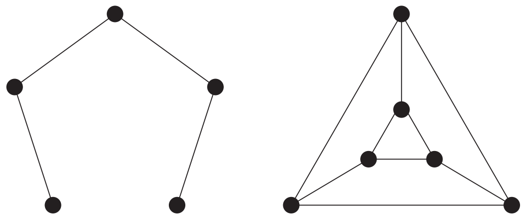


For example, the two cuts on the left are from $S_{T,3}$ and the cuts can be made in the order shown on the figures. The cut on the right, however is a cut into triangles of equal area but can't be constructed as described above and thus is not from $S_{T,3}$.

Problem 4. Edge Realizations of Graphs

1. Let G be a graph.

(a) Which of the graphs below can be edge realizations of some other graphs? **(0.5 point)**



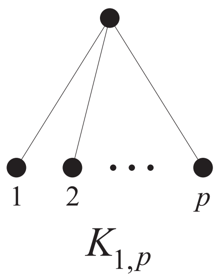
(b) Give the graph-theoretic definition of *a tree*. **(0.5 point)**

(c) Which trees are edge realizations? Of which graphs? **(1 point)**

2.

(a) Is it true that every graph without isolated vertices is edge realizable? Justify your answer. (Recall that a vertex is called *isolated* if there are no edges incident to it.) **(1 point)**

(b) Prove that for any integers $p, q \geq 1$ the disjoint union $K_{1,p} \cup K_{1,q}$ of two stars $K_{1,p}$ and $K_{1,q}$ with p and q edges respectively is edge realizable. Is the set of corresponding connected edge realizations finite or infinite? **(2 points)**



3. Let us denote by $\mathcal{N}$ the class of graphs G with the following property: for every pair of edges e_1 and e_2 of G the subgraphs induced by the edge neighborhoods of e_1 and e_2 are not isomorphic.

(a) Find a connected graph belonging to the class $\mathcal{N}$. **(2 points)**

(b) For which values of n there exist a connected graph with n vertices belonging to the class $\mathcal{N}$? **(3 points)**

Problem 5. Expansion in Graphs

1. Draw the Cayley graph on the group $\mathbb{Z}_6$ with the generating set $T = \{2, 3\}$. **(1 point)**.

2. The *spectral gap* of a connected undirected graph is the smallest positive eigenvalue of the Laplacian matrix. Find the spectral gap of the complete graph K_n on n vertices ($n \geq 3$). **(4 points)**.

3. Let $G_n = \mathrm{Cay}(S_n, Z_n)$ be the Cayley graph of the permutation group S_n ($n \geq 3$) with the generating set Z_n, where Z_n is a set consisting from $n-1$ transpositions. Define the graph $\mathcal{Z}_n$ on n vertices $\{1, 2, \ldots, n\}$ having the edge (i, j) if and only if $(i, j) \in Z_n$. Denote by $\mathcal{G}$ the set of all Cayley graphs $G_n = \mathrm{Cay}(S_n, Z_n)$, $n \geq 3$, such that the corresponding graph $\mathcal{Z}_n$ is a tree. Find all possible values of girth (the length of a shortest cycle) over all graphs from $\mathcal{G}$. **(5 points)**.

Problem 6. Generalized Commuting Graphs of Finite Groups

1.

(a) Let G be a group. Prove that $Z(G)$ is a subgroup of G. (That is prove that for any $g, h \in Z(G)$ we have $gh \in Z(G)$ and $g^{-1} \in Z(G)$). **(1 point)**

(b) Is there a group G such that its commuting graph $\Gamma(G)$ is of the from K_n for some $n \geq 2$? **(2 point)**

2. Determine whether the graph $\Gamma(D_6)$ is planar. (Recall that a graph is called *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.) **(3 points)**

3. Determine whether the graph $\Gamma(G)$ is connected, where $G = \mathrm{Sym}(\mathrm{Cube})$ is the group of all Euclidean symmetries of the cube in $\mathbb{R}^3$. **(4 points)**

Problem 7. Inhomogeneous Numbers

1. We define $a \oplus b := \frac{a+b}{2}$. Is $(\mathbf{R}, \oplus, \times)$ a commutative ring? **(1 point)**

2.

(a) Prove that for a commutative ring A we have: $x \in I_A \Leftrightarrow 4 - x \in I_A$. **(2 points)**
(b) Prove that, for every finite commutative ring A, we have $|I_A| \leq \frac{|A|+1}{2}$. **(2 points)**
(c) When does equality hold? **(2 points)**

3.

(a) Let A be a commutative ring. We suppose that $a, -a \notin I_A$. Prove that $A \backslash \{a, -a\}$ is a suffi-A. **(2 points)**
(b) Find a suffi-$\mathbb{N}$ with density $\frac{1}{2}$. (Recall that $\mathbb{N}$ does not include 0, even if we are in France). **(1 point)**

Problem 8. Mystic Powers of Two

1.

(a) Come up with a more general notion of an m-colouring and give a list of all 3-colourings of $P_1 = \{2, 4\}$, $P_2 = \{3, 5\}$ and their types. **(1 point)**

(b) Is it true that for any sequence s of length m there exists only finite number of initial data that admit an m-coloring of type s? **(2 points)**

2.

(a) Does there exist a sequence s of some length $m > 1$ such that there are n initial data which admit m-colouring of type s, where $m \equiv 0 \pmod 3$? **(2 points)**

(b) Let $s := (1, 2, 1)$. Find such $r \geq 1$ that $s^r := (1, 2, 1, 1, 2, 1, ..., 1, 2, 1)$ (r copies) is mystic. **(2 points)**

3. Is it true that $s = (9, 9, 9)$ is mystic and minimal? **(3 points)**

Problem 9. On Some Sequences Generated by a Function

1.

(a) Give an example of a function $f\colon [0,1] \to \mathbb{R}$ with unbounded variation on $[0,1]$. **(1 point)**

(b) Suppose f has bounded variation. Is it true that the limit $\lim_{x\to 1} f(x)$ always exists? **(1 point)**

2. Let $f(x) = \sin x$, and let $L(f,2) = \{(x_1,x_2) : \exists x \in (0,\infty) \text{ such that } x_1 = f(x), x_2 = f(2x)\}$.

(a) Is it true that $L(f,2) \supset \{(x_1,x_2) : x_1, x_2 \in (0,1)\}$? **(1 point)**

(b) Does there exist a continuous function $f\colon (0,\infty) \to \mathbb{R}$ such that $L(f,2) \supset \mathbb{R}\times\mathbb{R}$? **(3 points)**

3. Let $f(x) = x\sin x$, and let $L(f) = \{(x_n) : \exists x \in (0,\infty) \text{ such that } x_n = f(nx)\}$.

(a) Does the sequence of all prime numbers (p_n) belong to $L(f)$? **(2 points)**

(b) Does there exist a sequence (t_n) of natural numbers, such that for any $(x_n) \in L(f)$ the inequality $|t_n - x_n| \geq 1$ holds for almost all $n \in \mathbb{N}$ (that is for all n except for the finite number)? **(2 points)**

Problem 10. Pressing Coloured Buttons

1.

(a) Let X be a random variable taking values in $\{1, 2, \dots, n\}$ with $\mathbb{P}[X = i] = p_i$ for $1 \leq i \leq n$. Give a formula for *the expected value* of X. **(1 point)**

(b) Define $f, g, h \colon \mathbb{N} \to \mathbb{N}$ by $f(n) = 4^n - 2018n^2 + 10^{2018}$, $g(n) = 4^n + n^4$, $h(n) = 5^n$. Determine, with brief justification, which of f, g, h are asymptotically equivalent? **(1 point)**

2. Derive a closed-form expression for $A_{n,4}$. **(3 points)**

3. Let B_n be the number of ways for $a_1 = a_2 = \dots = a_n = t = 1$. Evaluate the limit $\lim_{n\to\infty} \sqrt[n]{\frac{A_{n,2n}}{B_{2n}}}$. **(8 points)**