9th International Tournament of Young Mathematicians

QUIZ

2 hours

- Each team (high school students only) is gathered in a separate room and works together. Written materials, electronics, literature or other sources are forbidden during the quiz, as well as any external help. Only brochures of the ITYM and paper language dictionaries are allowed.
- A solution for each of the 10 problems should be written **separately**.
- Indicate the **problem number** and page numbers on every solution.
- Please **don't** mention your country, team or other names anywhere.

Good luck!

Problem 1. Numbers from strings

1. Prove that the composition of increasing functions is increasing. **(1 point)**

2. Put the numbers below in ascending order:

$$a = 99^{99}$$
$$b = 9^{9^9}$$
$$c = 4^{4^{4^4}}$$
$$d = 3^{3^{3^{3^3}}}$$
$$e = 2^{2^{2^{2^{2^2}}}}$$

(4 points)

3. Suppose we interpret concatenated variables as a product. For example, the string "a=99,aaa" evaluates to $99 \times 99 \times 99$. Which is the minimal number n such that you can evaluate a string of lenght n to a number greater than $999 \ldots 99$ (n consecutive nines). Which is the maximal number you can obtained in this way? **(5 points)**

Problem 2. Monotonous Functions

1. a) Define what is a strictly increasing function at 0. **(1 point)**
b) Suppose f is strictly increasing and differentiable at 0. Find all possible values of $f'(0)$. **(1 point)**

2. Can such a function be not monotone on any interval $(-a, a)$ for any $a > 0$? **(3 points)**

3. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are two differentiable functions, satisfying the equation $f(x) \cdot g(x) = x$ on $\mathbb{R}$. Could they be:
a) Both convex on $\mathbb{R}$? **(2 points)**
b) Both strictly increasing and convex on $\mathbb{R}$? **(3 points)**

Problem 3. Dense analytic curves

1. Give the definition of an analytic function: $\mathbb{C} \to \mathbb{C}$. **(1 point)**

2. Find all complex solutions of the equations:
(a) $\exp(z) = i$; **(1 point)**
(b) $\exp(\exp(z)) = 1$. **(2 points)**
(c) $\exp(\exp(\exp(z)) = 1$ **(3 points)**

3. Let $c(t) = \exp(t(1+i))$ for real t. Is it true that, for all $\varepsilon > 0$, there is a point $P = m+ni \in \mathbb{C}$, where $m, n \in \mathbb{Z}$, such that the disc centered on P with radius ε intersects the image of c? **(3 points)**

Problem 4. Partitions of regular polygons

1. What is the maximal number of non-intersecting diagonals of an n-gon, such that no diagonal has its starting and end points more than k sides apart? **(2 points)**

2. Recall that P_n is a regular polygon with n sides inscribed in the unit circle and whose vertices are the n complex roots of unity. Denote by $\mathbb{A}_n^{(k)}$ the set of all k-angulations of P_n. If $A \in \mathbb{A}_n^{(k)}$, denote by $\chi(A)$ the length of the longest diagonal of A. Find the best possible lower bound for $\min_{A \in \mathbb{A}_n^{(k)}} \chi(A)$ and the best possible upper bound for $\max_{A \in \mathbb{A}_n^{(k)}} \chi(A)$. **(3 points)**

3. A dissection of P_n is by definition the union of the sides of P_n and of a collection of diagonals of P_n which do not cross (thus the minimal number of diagonals of a dissection is 0, in which case it is just P_n, and the maximal number of diagonals of a dissection is $n-3$, in which case it is a triangulation). Set $D_2 = 2$ by convention, and for $n \geq 3$ denote by D_n the number of dissections of P_n. Compute the value of

$$\sum_{n=2}^{\infty} D_n x^n$$

when x is small enough. **(5 points)**

Problem 5. Coin Toss Function

1. Let $X \in [a, b]$ be chosen uniformly. Give a definition of the expected value of $f(X)$, for a continuous function f, and calculate $\mathbb{E}[X^2]$. **(2 points)**

2. Fix $p \in (0, 1)$ and consider $x \in (0, 1)$.

(a) Show that $F_p\left(\frac{x+2}{8}\right) = AF_p(x)+B$ for some constants A, B, which you should determine. **(2 points)**

(b) Find $\int_0^1 F_p(x)^2 \, dx$. **(2 points)**

3. Find all real numbers k for which the inequality

$$(1+t)^k(1-t)^{1-k} \leq 1$$

holds for all $t \in (-1, 1)$. **(4 points)**

Problem 6. Rooks on graphs

1. a) Define the complete graph K_n. **(1 point)**
b) How many edges does it have? **(1 point)**

2. Take a 3-dimensional $m \times n \times p$ grid, where the rook can only move forwards, upwards, or rightwards. How many paths are there between the extremal vertices of the graph? **(3 points)**

3. Now suppose we have a regular infinite triangular grid with edges of lenght 1. How many paths are there between $(0, 0)$ and $(5, 0)$ with length 12? **(5 points)**

Problem 7. Weighted sums of distances

Let $\triangle ABC$ be a triangle and let P be a point in its plane

1. Define the barycentric coordinates of P in terms of $\triangle ABC$. **(1 point)**.

2. Let P be a point with barycentric coordinates (λ, μ, ν) with respect to $\triangle ABC$. Take the lines AP, BP and CP, and reflect them on the bissectors AI, BI and CI respectively (I is the incenter). Prove that all these lines pass through the point Q with barycentric coordinates $\left(\frac{BC^2}{\lambda}, \frac{CA^2}{\mu}, \frac{AB^2}{\nu}\right)$ with respect to $\triangle ABC$. (This is called the *isogonal conjugate of* P.) **(4 points)**.

3. Consider the function $F(X) = \lambda d_1^2 + \mu d_2^2 + \nu d_3^2$ where $\lambda, \mu, \nu \in \mathbb{R}^+$. Prove that the minimum of F is attained at the point P' which is the isogonal conjugate of $P(\lambda, \mu, \nu)$ with respect to $\triangle ABC$. **(5 points)**.

Problem 8. A communication network

1. Give the definition of the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$. **(1 point)**

2. a) Find all connected graphs G with $n \geq 2$ vertices such that $s(G) = n$. **(2 points)**

b) The *complement* $\overline{G}$ of a graph G is that graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. Is is true that if a graph G with $n \geq 2$ vertices is disconnected, then $s(\overline{G}) \in \{n, n+1\}$? **(3 points)**

3. A *cubic tree* T_n is a tree with $n \geq 2$ vertices, all of which have either degree 3 or degree 1. Determine (with proof) a formula for $s(T_n)$. **(4 points)**

Problem 9. An Approximation Problem

1. Show that the set of rational points $\mathbb{Q}^2$ is dense in the real plane $\mathbb{R}^2$. **(2 points)**

2. Given $a, b, c \in \mathbb{R}$, consider the following subset of real numbers

$$U = U(a,b,c) = \left\{ \frac{ax^2 + bx + c}{x} \mid x \in \mathbb{Q},\, x \neq 0 \right\}.$$

Determine all $a, b, c \in \mathbb{R}$ for which the set U is dense in the real line. **(4 points)**

3. Let ABC be a triangle in the real plane and $\varepsilon > 0$ a real number. Prove that there exists a triangle PQR with *rational* side lenghts such that the distances from the vertices P, Q, R to the vertices A, B, C respectively are less than ε. **(4 points)**

Problem 10. Kinds of convexity

1. For what s the function $f(x) = \begin{cases} 2-x, & x \le 1, \\ 3x-2, & 1 < x \le 2, \\ 10-3x, & 2 < x < 3, \\ x-2, & x \ge 3 \end{cases}$ is s-convex in second sense? **(2 points)**

2. a) Let f is strictly convex or strictly concave on I and u, u_1, v, $v_1 \in I$ such that $u+v = u_1+v_1$. Prove that

$$f(u) + f(v) = f(u_1) + f(v_1) \quad \Longleftrightarrow \quad u = u_1 \text{ or } u = v_1.$$ (**2 points**)

b) Solve the equation

$$\sqrt[4]{1-\sin^4 x} + \sqrt[4]{1-\cos^4 x} = \sqrt[4]{12}.$$ (**2 points**)

3. For $\alpha \neq 0$, a function $f : I \to (0; +\infty)$ is said to be α-convex if

$$f(\lambda x + (1-\lambda)y) \le (\lambda f(x)^\alpha + (1-\lambda) f(y)^\alpha)^{\frac{1}{\alpha}}$$

for all $x, y \in I$ and $\lambda \in [0; 1]$.

a) Using the limit define α-convex function for $\alpha = 0$. **(1 points)**

b) Using the limit define α-convex function for $\alpha \to +\infty$. **(1 points)**

c) Let $0 < a_1 \le a_2 \le \ldots \le a_n$, $0 < b_1 \le b_2 \le \ldots \le b_n$. For what α the inequality

$$\prod_{i=1}^{n} f(a_i + b_i) \ge \prod_{i=1}^{n} f(a_i + b_{n-i+1})$$

holds for any α-convex function f on $(0; +\infty)$? **(2 points)**