8th International Tournament of Young Mathematicians

QUIZ

2 hours

- Each team (high school students only) is gathered in a separate room and works together. Written materials, electronics, literature or other sources are forbidden during the quiz, as well as any external help. Only brochures of the ITYM and paper language dictionaries are allowed.
- A solution for each of the 10 problems should be written **separately**.
- Indicate the **problem number** and page numbers on every solution.
- Please **don't** mention your country, team or other names anywhere.

Good luck!

Problem 1. Rankings of Teams

1. a) Give the list of all the elements of $\mathcal{R}_3$. **(1 point)**
b) Give the list of all the elements of $\mathcal{P}(\sigma)$ for $\sigma = (4, 1, 3, 2)$. **(1 point)**
c) Let X be a random element of the set $\{1, \dots, n\}$ chosen with the following distribution: the probability of appearance of the value $k \in \{1, \dots, n\}$ is p_k. Give a formula for the expectancy $\mathbb{E}[X]$. **(1 point)**

2. Show that $\frac{\mathbb{E}[\max(\rho_n)^2]}{n^2}$ converges to a positive value as $n \to \infty$. **(3 point)**

3. Compute, for fixed integers $i, j \geq 0$, the limiting value of $\mathbb{P}[f_1(\rho_n) = i \text{ and } f_2(\rho_n) = j]$ as $n \to \infty$. **(4 point)**

Problem 2. Cutting Segments

1. Let D be a domain in the real plane. Give a definition for D to be *bounded.* **(1 point)**

2. Let $\mathcal{A} = A_1 \dots A_n$ and $\mathcal{B} = B_1 \dots B_n$ be two convex n-gons in the plane such that $\mathcal{B} \subset \mathcal{A}$. Show that the number of segments A_iB_j that contain an inner point of $\mathcal{B}$ is strictly less than $(n-1)^2$. **(3 points)**

3. Let $\mathcal{A} = A_1 \dots A_{2016}$ be a convex polygon. One drew all its diagonals. They divided $\mathcal{A}$ into smaller polygons.
a) Show that each of these smaller polygons has at most 2015 sides. **(3 points)**
b) Give an example of $\mathcal{A}$ for which there is a smaller polygon $\mathcal{B} = B_1 \dots B_{2015}$ with 2015 sides. Calculate the number of cutting segments A_iB_j. **(3 points)**

Problem 3. Loose Number Sets

1. Give a definition of the limit of a sequence of real numbers $\{x_n\}$. Include also the definition of $\lim_{n\to\infty} x_n = +\infty$. **(1 point)**

2. a) Let $Q(N) = n$. Prove that $n(n-1)/2 < N$. **(2 points)**

b) Let $P(N) = n$. Prove that $n(n-1)/2 < 2N - 1$. **(2 points)**

3. Let $x_1, \dots, x_n$ be n different positive integers strictly less that 2016 such that none of them is divisible by 13, and none of the partial sums of these numbers (including the sum of all numbers) is divisible by 13. Assuming that n can be arbitrary, what is the maximal possible sum of these numbers? **(5 points)**

Problem 4. Suprema of Integer Polynomials

1. Give a definition of the *supremum* of a set of real numbers. (**1 point**)

2. a) Prove that for all $r \geq 0$, one has $c(0, r) = r$. (**2 points**)
b) Prove that if p and q are coprime integers and $0 \leq r \leq \frac{1}{q^2}$, then $c(\frac{p}{q}, r) = qr$. (**2 points**)
c) Prove that $\frac{1}{2} \leq c(\frac{1}{2}, \frac{1}{2}) \leq \frac{1}{\sqrt{2}}$. (**2 points**)

3. Prove that if $c(\frac{1}{4}, \frac{1}{4}) > \frac{1}{4}$ then $c(\frac{1}{2}, \frac{1}{2}) > \frac{1}{2}$. (**3 points**)

Problem 5. Composite Polygons

1. Let E be a convex polygon, P a point on a side of E and l a direction in the real plane. Construct a sequence of points $x_1, x_2, \dots$ recursively according to the following rules.

- $x_1 = P$,
- all points x_k lay on the boundary of the polygon E,
- the segment $[x_1, x_2]$ is parallel or orthogonal to l,
- each segment $[x_k, x_{k+1}]$ is orthogonal to $[x_{k-1}, x_k]$.

If we reach a point x_n with one of the following properties, we stop the construction:

- $x_n = x_k$ for $k < n$,
- x_n is a vertex of E,
- $[x_{n-1}, x_n]$ is orthogonal to the side of E containing x_n.

Otherwise, the sequence is infinite.
This sequence is call a *trajectory* of P in direction l.

a) Give one example of polygon E, point P and direction l with a finite trajectory. Then give an example with infinite trajectory. (**1 point**)
b) Give a criteria of divisibility for a polygon E in terms of the trajectories of the vertices of E. (**2 point**)

2. Let E be a parallelogram with vertices $ABCD$ such that any trajectory of any point P on the side AB which is not a vertex consists of exactly 4 points x_1, x_2, x_3, x_4 whith $x_2 \in BC$ and $x_3 \in CD$. Is it true that $AB = BC = CD = DA$? (**3 point**)

3. Find an example of a convex quadrilateron with the same property of trajectories as in the question 2, which is not a rhombus (*i.e.* not all sides are equal). (**4 point**)

Problem 6. Different Means

1. a) Give a definition of the Taylor series of a function $f : \mathbb{R} \to \mathbb{R}$ infinitely differentiable at a point a. (**1 point**)
b) Find a Taylor series expansion for $x \mapsto \sqrt{x}$ at the point $x = 1$. (**1 point**)

2. Consider two positive symmetric functions $M(a,b)$ and $N(a,b)$ with a domain $\mathbb{R}_+ \times \mathbb{R}_+$. Proof that if $M(a,b)$ and $N(a,b)$ are continuous and for all $a < b$ holds

$$a < M(a,b) < b \text{ and } a < N(a,b) < b,$$

then $M \otimes N(a,b)$ exists. (**3 points**)

3. Find rational functions $r(t)$ and $q(t)$ such that the function $u : t \mapsto (A \otimes G(1-t, 1+t))^{-1}$ is a solution to the following differential equation:

$$u''(t) + r(t)u'(t) + q(t)u(t) = 0.$$

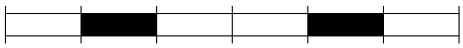
Hint: use the representation of $u(t)$ as an integral. (**5 points**)

Problem 7. A Colouring Game

1. Give a formal definition of the *maximal gain* for a player. **(1 point)**

2. a) After 2 turns of the game on the segment $[0,1]$, the set of black points is the following:

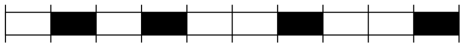
$$I = \left[\frac{1}{6}, \frac{2}{6}\right] \cup \left[\frac{4}{6}, \frac{5}{6}\right]$$



Show that for any $n \in \mathbb{N}$, Alice's maximal gain is at least $\frac{1}{3}$ and at most $\frac{2}{3}$. **(2 points)**

b) After 4 turns of the game on the segment $[0,1]$, the set of black points is the following:

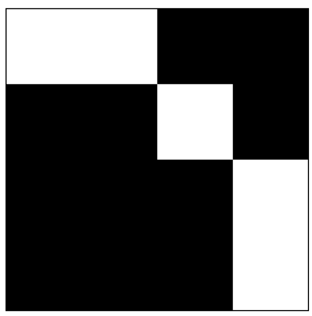
$$I = \left[\frac{1}{10}, \frac{2}{10}\right] \cup \left[\frac{3}{10}, \frac{4}{10}\right] \cup \left[\frac{6}{10}, \frac{7}{10}\right] \cup \left[\frac{9}{10}, \frac{10}{10}\right]$$



Show that for any $n \in \mathbb{N}$, Alice's maximal gain is at least $\frac{2}{5}$ and at most $\frac{3}{5}$. **(2 points)**

3. We play the same game on a 1×1 square, initially white. At each turn, one player chooses $t \in [0,1]$ and the other player places a $t \times t$ square inside the initial square, with sides parallel to those of the initial square. Alice's gain G_A is area of the white part. We play n such turns.

In the example below, two squares with sides $\frac{1}{2}$ and $\frac{3}{4}$ were placed. $G_A = \frac{5}{16}$.



Show that for any integer n, Alice has a strategy to ensure:

a) $G_A \geq \frac{1}{8}$. **(2 points)**

b) $G_A \geq \frac{1}{4}$. **(3 points)**

Problem 8. Functional Equations

1. Give a definition for a function $f : \mathbb{R} \to \mathbb{R}$ to be:

a) *continuous.* **(1 point)**
b) *analytic.* **(1 point)**

2. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Suppose that $f(f(x)) = 0$ for all $x \in \mathbb{R}$. Can f be surjective from $\mathbb{R}$ to $[0, \infty)$ and:

a) continuous? **(1 point)**
b) infinitely differentiable? **(2 points)**
c) analytic? **(1 point)**

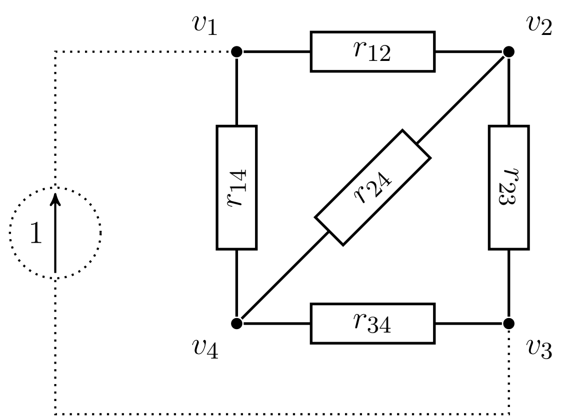
3. Suppose f is an arbitrary continuous function satisfying the equation $f(f(x)) = x^2$ on $\mathbb{R}$.

a) Can f take negative values? **(1 point)**
b) Is f necessarily strictly increasing on $[0, \infty)$? **(1 point)**
c) Can f be two (or more) times differentiable at the point $x = 0$? **(2 points)**

Problem 9. Mathematics of Electric Circuits

1. a) Give a definition of a *generating set* of a group G. **(1 point)**
b) Give an example of a generating set of two element for the symmetric group S_n. **(1 point)**

2. a) Find the resistance of the electric circuit on the figure bellow. **(3 points)**



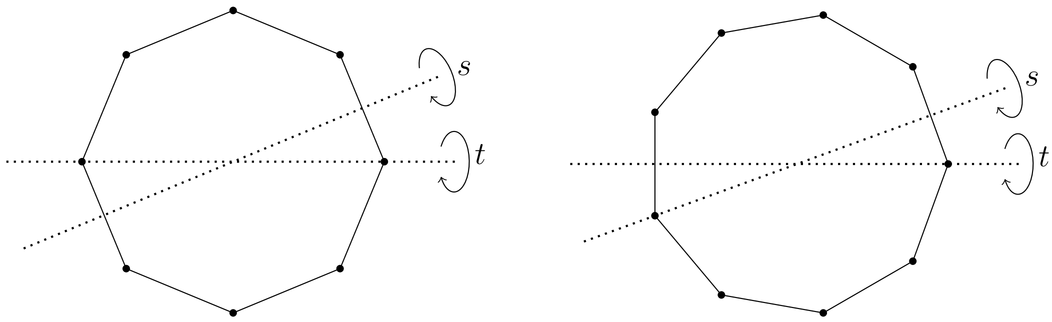
b) Under which conditions on r_{12}, r_{23}, r_{34}, r_{14} the electric current between vertices v_2 and v_4 is zero? **(1 point)**

3. Let s and t be the reflections with respect to two adjacent symmetry lines of a regular n-gon (as shown in the picture below). Prove that they generate the entire symmetry group of the n-gon and find the Cayley graph of the group associated with $\{s, t\}$. **(4 points)**

Problem 10. Rich Sequences

1. a) Let $M = (m_{i,j})$ be an $n \times n$ matrix. Give a formula for the determinant of M. **(1 point)**
b) Prove that if two columns of a matrix are equal, then its determinant is zero. **(1 point)**

2. Find all rich sequences when $S = \mathbb{Z}_3$ is the set of all residues modulo 3. **(4 points)**



3. Let $(a_i)_{i\geq 1}$ be a sequence of integers. Define a new sequence $(b_i)_{i\geq 1}$ as:

$$\begin{aligned} b_1 &= a_1, \\ b_2 &= a_1 + a_2, \\ b_3 &= a_1 + 2a_2 + a_3, \\ b_4 &= a_1 + 3a_2 + 3a_3 + a_4, \\ &\dots \end{aligned}$$

where in general $b_{k+1} = \sum_{i=0}^{k} \binom{k}{i} a_{i+1}$. Prove that for every $n \in \mathbb{N}$, the determinants $|H_{1,n}|$ of $(1, n)$-order Hankel matrices for sequences $(a_i)_{i\geq 1}$ and $(b_i)_{i\geq 1}$ are equal. **(4 points)**