PROBLEMS FOR THE 8^{th} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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Contents

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Notation

$\mathbb{N} = \{1, 2, 3, \ldots\}$	set of natural numbers (positive integers)
$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$	ring of residue classes modulo n
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	sets of integer, rational, real and complex numbers
$\mathbb{R}_+$	set of positive real numbers
$\mathbb{F}_q$	finite field of size $q = p^k$
$M \otimes N$	mean-product of two functions M and N
$\gcd(x_1, \ldots, x_n)$	greatest common divisor of $x_1, \ldots, x_n$
$\mathbb{P}[.]$ and $\mathbb{E}[.]$	probability and expectation
$\det(A)$ or $\|A\|$	determinant of a matrix A
$\|x\|$	absolute value of x
$\#M$ or $\|M\|$	cardinality of a set M
$\sup\limits_{x \in M} f(x)$	supremum of a set $\{f(x) \mid x \in M\}$

1. Rankings of Teams

A mathematics tournament is being held in Saint Petersburg with $n \in \mathbb{N}$ participating teams $T_1, \ldots, T_n$. In the end of the tournament, each team will be ranked according to its rating (teams can have the same rating). The teams with the highest rating will have rank 1, the teams with the second highest rating will have rank 2, and so on. Denote by $\mathcal{R}_n$ the set of all possible rankings. In other words,

$$\mathcal{R}_n = \left\{(r_1, \ldots, r_n) \in \mathbb{N}^n \;\middle|\; \text{for any integer } 1 \le t \le \max_k(r_k), \text{ there exists } i \text{ such that } r_i = t\right\},$$

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where r_i is the rank of the team T_i.

1. Let a_n be the number of elements in $\mathcal{R}_n$ and put $a_0 = 1$. Find a recurrent formula for a_n in terms of $a_0, a_1, ..., a_{n-1}$.

2. Let $\sigma = (s_1, \dots, s_n)$ be a permutation of $\{1, \dots, n\}$. Find the cardinality of the subset

$$\mathcal{P}(\sigma) := \{(r_1, \dots, r_n) \in \mathcal{R}_n \mid \text{for any } 1 \le i < j \le n, \text{ one has } r_i < r_j \text{ if and only if } s_i < s_j\}.$$

3. Show that for every real number x with $|x|$ sufficiently small, one has

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \frac{1}{2 - e^x}.$$

4. Denote by M_n the sum of all possible last ranks, that is

$$M_n = \sum_{(r_1,\dots,r_n) \in \mathcal{R}_n} \max(r_1, \dots, r_n)$$

and let $M_0 = 0$. Compute the series $\sum_{n=0}^{\infty} \frac{M_n}{n!} x^n$.

5. Let $\rho_n = (X_1, \dots, X_n)$ be a random element of $\mathcal{R}_n$ chosen uniformly. Study the behaviour of $\mathbb{E}[\max(\rho_n)]$, that is the expected value of the rank of a team with the lowest rating, as $n \to \infty$.

6. Study the behaviour of $\mathbb{E}[X_1]$, that is the expected value of the rank of the team T_1, as $n \to \infty$.

7. Denote by $f_t(\rho_n)$ the number of teams with rank t: $f_t(\rho_n) = \#\{1 \le i \le n \mid X_i = t\}$.

a) Let k be a positive integer. Find $\lim_{n\to\infty} \mathbb{P}[f_1(\rho_n) = k]$, that is the limit of the probability that exactly k teams have rank 1.

b) How does $\mathbb{E}[f_1(\rho_n)]$ behave as $n \to \infty$ (the expected number of teams ranked first)?

c) How does $\mathbb{P}[X_1 = 1]$ behave as $n \to \infty$ (the probability that the team T_1 is first)?

8. Same questions for $t = 2$, 3, etc.

9. Denote by $(X'_1, \dots, X'_n)$ the coordinates of ρ_n ordered in non-increasing order. Let $s \in]0,1[$ be a fixed real number. Study the behaviour of the random variable $X'_{\lfloor sn \rfloor}$ as $n \to \infty$, where $\lfloor y \rfloor$ denotes the integer part of a real number y.

10. Study other properties of ρ_n. Suggest and investigate additional directions of research.

2. Cutting Segments

Let $\mathcal{A}$ and $\mathcal{B}$ be two bounded domains in the real plane. We suppose that either $\mathcal{B} \subseteq \mathcal{A}$ or $\mathcal{B} \cap \mathcal{A} = \varnothing$. For positive integers m and n, let $A_1, \dots, A_m$ and $B_1, \dots, B_n$ be some points on the boundaries of $\mathcal{A}$ and $\mathcal{B}$ respectively. We say that a segment A_iB_j is *cutting* if it contains an inner point of the domain $\mathcal{B}$ (in other words, it is dividing $\mathcal{B}$ into parts, at least two of which have nonzero area). Denote by $Cut_{\mathcal{B}}(A_1 \dots A_m, B_1 \dots B_n)$ the set of cutting segments. We would like to find all possible values of the cardinality of this set, and in particular its minimum and maximum, in the following cases:

(i) the points (A_i) and (B_j) are fixed on the respective boundaries, and the domain $\mathcal{B}$ can float (move and rotate) strictly inside the domain $\mathcal{A}$;

(ii) the points (A_i) and (B_j) are fixed on the respective boundaries, and the domain $\mathcal{B}$ can float (move and rotate) strictly outside the domain $\mathcal{A}$;

(iii) the domains $\mathcal{A}$ and $\mathcal{B}$ are fixed as well as the points (A_i) on the boundary of $\mathcal{A}$, and the points (B_j) can float on the boundary of $\mathcal{B}$.

1. Let $\mathcal{A} = A_1 \dots A_m$ be a regular m-gon and $\mathcal{B} = B_1 \dots B_n$ be a regular n-gon.

a) What area may the polygon $\mathcal{B}$ have so that it could be placed strictly inside $\mathcal{A}$?
b) Suppose $\mathcal{B}$ lies strictly inside $\mathcal{A}$. What minimum and maximum number of cutting segments can there be (*cf.* Figure 1)?
c) Determine $Cut_{\mathcal{B}}(A_1 \dots A_m, B_1 \dots B_n)$ in the case (i), depending on the size and the position of $\mathcal{B}$ within $\mathcal{A}$.
d) Consider the case (ii).
e) Consider the case (iii).

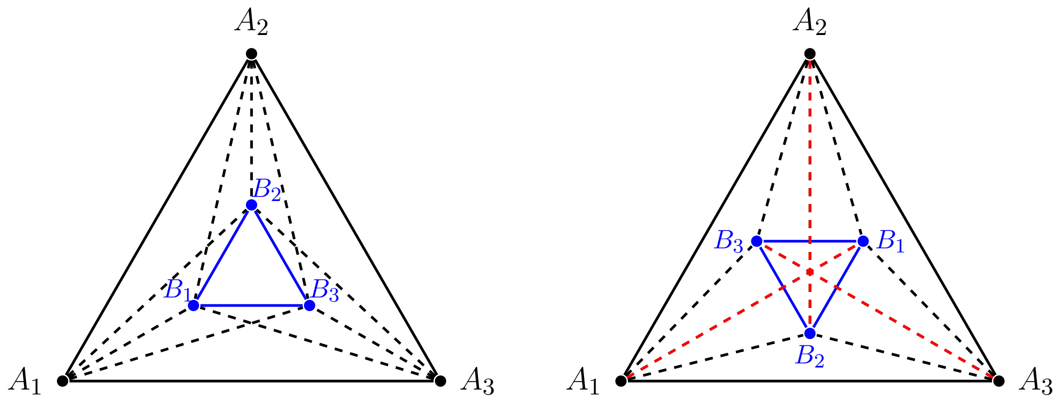

FIGURE 1. Two examples where $\mathcal{A}$ and $\mathcal{B}$ are triangles in the case (i), and $|Cut_{\mathcal{B}}(A_1A_2A_3, B_1B_2B_3)|$ is equal to 0 and 3.

2. Study the problem for other convex polygons.

3. Study the problem for non-convex polygons. Start with $\mathcal{A}$ convex and $\mathcal{B}$ non-convex.

4. Study the problem when $\mathcal{A}$ and $\mathcal{B}$ are circles. Start with the case (i) and suppose that the points divide the respective circumferences into arcs of equal length (equidistribution).

5. Study the problem when $\mathcal{A}$ and $\mathcal{B}$ are ellipses.

6. Generalise to higher dimensions $N \geq 3$. In particular, study the problem when $\mathcal{A}$ and $\mathcal{B}$ are

a) N-dimensional simplexes;
b) N-dimensional balls.

7. Suggest and investigate other directions of research.

3. **Loose Number Sets**

We say that a set of integers is *loose*, if it does not contain four different numbers $a < b < c < d$ such that $a + d = b + c$. Denote by $P(n)$ the maximal number of elements in a loose subset of the set $\{1, 2, \dots, n\}$. We are interested in calculating $P(n)$ for large n or finding lower and upper bounds.

1. Compute $P(n)$ for small n up to 10.

2. Prove that $\lim\limits_{n\to\infty} \frac{P(n)}{n} = 0$.

3. Try to find best possible upper and lower bounds for $P(100)$ and $P(2016)$.

4. Is it true that $\lim\limits_{n\to\infty} \frac{P(n)}{n^{\varepsilon}} = 0$ for

a) $\varepsilon = 1/2$,

b) $\varepsilon = 1/3$,
c) any $\varepsilon > 0$?

5. We call a sequence *very loose* if it contains neither four different numbers $a < b < c < d$ such that $a + d = b + c$, nor triples $a < b < c$ such that $a + c = 2b$. Denote by $Q(n)$ the maximal number of elements in a very loose subset of the set $\{1, 2, \ldots, n\}$. How the answers to the above questions change for $Q(n)$?

6. Suggest and investigate generalisations.

Note. *If you use computer programs, then please provide the full source codes and all necessary instructions so that other participants and the jury members are able to reproduce your results.*

4. Suprema of Integer Polynomials

Let $D(a, r) := \{z \in \mathbb{C} \mid \mid z - a \mid \leq r\}$ be the closed disk of radius $r \in \mathbb{R}_+$ centred at $a \in \mathbb{C}$ in the complex plane. Denote by $c = c(a, r)$ the largest real number such that for any nonzero polynomial $P \in \mathbb{Z}[z]$ with integer coefficients, one has

$$\sup_{z \in D(a,r)} |P(z)| \geq c^n,$$

where n is the degree of P.

1. Find $c(\frac{1}{2}, \frac{1}{2})$ or give lower and upper bounds.

2. Find $c(a, r)$ or give lower and upper bounds, when $a \in \mathbb{Q}$.

3. Estimate $c(a, r)$ when a is a quadratic number, that is $ua^2 + va + w = 0$ for some $u, v, w \in \mathbb{Z}$ and $u \neq 0$.

4. Study the general case.

5. Suggest and study additional directions of research.

5. Composite Polygons

Let ℓ be a line in the real plane. We will say that a polygon P is *divisible by* ℓ if there exist two finite families of lines F_1 and F_2 in the plane such that

- the lines in F_1 are parallel to ℓ,
- the lines in F_2 are orthogonal to ℓ,
- the polygon P is divided by F_1 and F_2 into parts – each part is either a rectangular triangle or a rectangle.

We will also say that P is *divisible by a segment* s if it is divisible by the line containing s. Finally, a polygon will be called *composite* if it is divisible by at least one line.

1. Given a triangle, describe all lines by which it is divisible.

2. Given a convex quadrilateral, describe all lines by which it is divisible.

3. Give a criterion for a non-convex quadrilateral to be composite.

4. A polygon is *obtuse* if all its angles are greater than $\pi/2$. Does there exist an obtuse convex pentagon which is

a) not divisible by at least one of its sides?
b) not divisible by all its sides?
c) not composite?

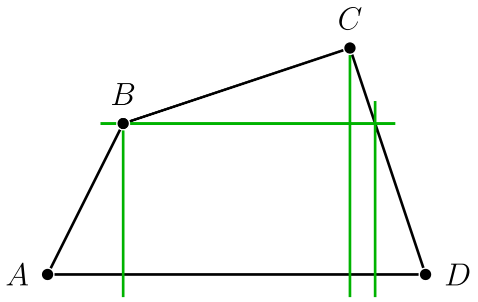

FIGURE 2. A convex quadrilateral $ABCD$ divisible by its side AD.

5. In how many parts can a composite obtuse convex pentagon be divided by a line ℓ, if

a) ℓ contains a side?
b) ℓ is neither parallel nor orthogonal to any of the sides?

6. Study the divisibility of non-convex and non-obtuse pentagons.

7. Consider other polygons. Describe composite n-gons when $n > 5$.

8. Suggest and study additional directions of research. One could also try to extend the notion of divisibility to α-*divisibility* where α is the angle between families F_1 and F_2.

6. **Different Means**

Let M and N be two real-valued functions defined in the domain $\mathbb{R}_+ \times \mathbb{R}_+$. Given any positive real numbers a and b, one can construct two sequences (a_n) and (b_n) satisfying the following recurrence relations

$$a_n = M(a_{n-1}, b_{n-1}) \quad \text{and} \quad b_n = N(a_{n-1}, b_{n-1}),$$

where $a_0 = a$, $b_0 = b$ and $n \in \mathbb{N}$. If the limits $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist and coincide for all $a, b \in \mathbb{R}_+$, then the function

$$M \otimes N(a, b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

is well-defined in $\mathbb{R}_+ \times \mathbb{R}_+$. We will call it the *mean-product* of M and N.

1. Consider two classical functions – the arithmetic mean $A(a, b) = \frac{a+b}{2}$ and the harmonic mean $H(a, b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}$. For any $a, b \in \mathbb{R}_+$ and $n \in \mathbb{N}$, take $a_n = A(a_{n-1}, b_{n-1})$ and $b_n = H(a_{n-1}, b_{n-1})$ with $a_0 = a$ and $b_0 = b$.

Prove that the sequences (a_n) and (b_n) converge to the same limit. Determine $A \otimes H$. Is it a symmetric function, that is $A \otimes H(a, b) = A \otimes H(b, a)$?

2. Let $G(a, b) = \sqrt{ab}$ be the geometric mean.

a) Show that the functions $A \otimes G$ and $H \otimes G$ are well-defined in $\mathbb{R}_+ \times \mathbb{R}_+$.
b) Prove that

$$A \otimes G(a, b) = \pi \left(2 \int_0^\infty \frac{1}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} \, dx \right)^{-1}.$$

c) Find a formula for $H \otimes G$.

3. Consider a Hoelder mean

$$H_p(a, b) = \sqrt[p]{\frac{a^p + b^p}{2}},$$

where $p \neq 0$ is an arbitrary real number. Investigate the following functions (if they are well-defined, then find a formula or a Taylor expansion):

a) $H_p \otimes G$;
b) $H_p \otimes H_q$, for any real nonzero p and q;
c) $H_p[\lambda] \otimes H_q[\mu]$, where $0 < \lambda, \mu < 2$ and $H_p[t](a,b) = \sqrt[p]{\frac{t \cdot a^p + (2-t)\cdot b^p}{2}}$.

4. One can instead construct the sequences (a_n) and (b_n) as follows:

$$a_n = M(a_{n-1}, b_{n-1}) \quad \text{and} \quad b_n = N(a_n, b_{n-1}),$$

where $n \in \mathbb{N}$, $a_0 = a$ and $b_0 = b$. If the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and coincide for all positive a and b, then one can define the *Archimedes mean-product* of M and N,

$$M \otimes_{\mathcal{A}} N(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Investigate the following functions (if well-defined, find a formula or a Taylor expansion):

a) $H_p \otimes_{\mathcal{A}} H_p$;
b) $H_p \otimes_{\mathcal{A}} G$;
c) $H_p \otimes_{\mathcal{A}} H_q$;
d) $H_p[\lambda] \otimes_{\mathcal{A}} H_q[\mu]$, where $0 < \lambda, \mu < 2$.

5. Consider other means: Lehmer means $\left(\frac{a^p+b^p}{a^{p-1}+b^{p-1}}\right)^{\frac{1}{p}}$, Stolarsky means $\left(\frac{a^p-b^p}{p(a-b)}\right)^{\frac{1}{p-1}}$, weighted versions of means and their convex combinations.

6. Find criteria for a mean-product to be continuous, symmetric, smooth, analytic, homogeneous, etc.

7. Try to generalise results to means of more than two variables.

7. A Colouring Game

Alice and Bob play a game on a circle of circumference 1. Initially, the circle is completely white. In each step, one player picks a real number $t \in [0,1]$ and the other chooses an arc of length t. All white points in the chosen arc become black and all black points become white. The players pick numbers alternately, starting with Alice. The game stops after n steps, where $n \in \mathbb{N}$ is given in advance. In the end, Alice's gain, denoted by G_A, is the proportion of points which are white. Bob's gain, denoted by G_B, is the proportion of points which are black.

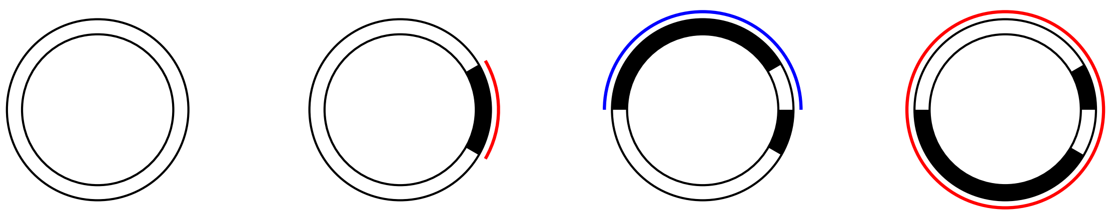

FIGURE 3. An example with $n = 3$. Alice picks $t = 1/6$ and Bob chooses an arc on the right. Then he picks $t = 1/2$ and Alice chooses an arc on the top. Finally Alice picks $t = 1$ and Bob chooses the entire circle. Here $G_A = G_B = \frac{1}{2}$.

1. Depending on n, does there exist a strategy allowing Alice to ensure a gain at least $\frac{1}{2}$? And what about Bob?

2. What maximal gain can Alice ensure? Start with $n = 2, 3, 4$.

In the next two questions, the game is on the segment $[0,1]$ which is initially white. In each step, a player picks a real $0 \leq t \leq 1$ and the other player chooses a subinterval of length t

within $[0, 1]$. The white points in the subinterval become black and vice-versa. The gains are defined in a similar manner.

FIGURE 4. An example with $n = 3$. Alice picks $t = 1/3$ and Bob chooses a subinterval on the left. Then he picks $t = 1/2$ and Alice chooses a subinterval on the right. Finally, Alice chooses $t = 1$ and Bob chooses the entire segment $[0, 1]$. As a result $G_A = G_B = \frac{1}{2}$.

3. What is the maximal gain that Alice can ensure (as a function of n)?

4. Find the maximal gains that Alice and Bob can ensure if they are only allowed to pick real numbers t which are

a) of the form $\frac{1}{k}$ with $k \in N$;
b) not greater than $\frac{1}{2}$.

5. Now, the game is on the segment $[0, 1]$ with 3 colours: white, blue and red. All white points in a chosen subinterval become blue, all blue points become red and all red points become white. Bob's gain is the proportion of white and blue points, Alice's gain is the proportion of red points. Find the maximal gain that Alice can ensure after n steps.

6. Suggest and study additional directions of research.

8. **Functional Equations**

Let $]a, b[$ be a non-empty open interval of the real line with $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. We would like to investigate equations of the form

$$f(f(x)) = g(x),$$

where f and g are two real-valued functions defined on the interval $]a, b[$.

1. Given that $g(x) = e^x$, does the equation have a solution f which is

a) continuous?
b) differentiable (twice differentiable, etc.)?
c) analytic?
d) presentable as a power series convergent in $]a, b[$?

Start with the case $]a, b[= \mathbb{R}$.

2. Analoguous questions when

a) $g(x) = x^2 + px + q$ with $p, q \in \mathbb{R}$;
b) $g(x)$ is a polynomial of degree $n > 2$.

3. Solve the equation $f(f(x)) = f'(x)$ for a continuously differentiable function $f :]a, b[\to \mathbb{R}$.

4. Investigate the equations $f(f(x)) = \int_a^b f(t)dt$ and $f(f(x)) = \int_a^x f(t)dt$, where $f : [a, b] \to \mathbb{R}$ is a) continuous; b) Riemann integrable; c) Lebesgue integrable.

5. Let M be a bounded closed interval $[a, b]$ or the entire $\mathbb{R}$. We say that a continuous function $g : M \to \mathbb{R}$ is a *k-square* if there exists a continuous function $f : M \to \mathbb{R}$ such that $g(x) = f(f(x))$ for all $x \in M$. Determine the density of k-squares in the set $C(M)$ of continuous functions on M for different topologies.

6. Suggest and study other directions of research.

9. **Mathematics of Electric Circuits**

An electric circuit can be viewed as an undirected weighted graph without loops and with multiple edges, where the weight of each edge is the resistance of the corresponding circuit element. Given such graph Γ, we are interested in measuring the resistance R_{ij} between any pair of vertices v_i and v_j of the graph. A well-known method consists in connecting a unit current source between the vertices, and applying Kirchhoff's circuit laws and Ohm's law in order to find the potentials at all vertices (see Figure 5). The resistance R_{ij} will then be the difference of potentials at v_i and v_j.

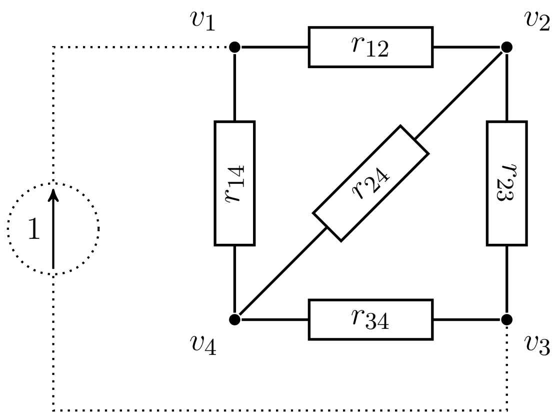

FIGURE 5. Connecting a unit current source between the vertices v_1 and v_3.

1. Let r_{ij} be the resistance (weight) of the edge connecting vertices $v_i, v_j \in \Gamma$. We denote $r_{ij} = \infty$ (or rather $1/r_{ij} = 0$) if there is no edge between these two vertices. Using Kirchhoff's laws define a system of linear equations that allows to compute the resistance R_{ij} between any two vertices v_i, v_j in terms of the weights r_{ij}.

2. Assume that the graph is built from the edges of a polyhedron in an N-dimensional Euclidean space and the resistance of any edge is exactly 1 ohm. Compute the resistance between opposite vertices for the following polyhedrons:

a) a regular polygon in the real plane;
b) a cube in the 3-dimensional Euclidean space;
c) an octahedron in the 3-dimensional Euclidean space;
d) an icosahedron in the 3-dimensional Euclidean space;
e) a cube in the N-dimensional Euclidean space.

3. Assume that you can vary the resistance of each edge in a given interval $[a, b]$. What are the maximal and minimal resistances that you can get between two opposite vertices for the above graphs?

4. Let S_n be the symmetric group of degree n, and let $T \subset S_n$ be the set of transpositions $(1\,2)$, $(2\,3)$, …, $(n{-}1\,n)$. The Cayley graph associated with (S_n, T) is defined as the undirected graph having $n!$ vertices corresponding to the permutations in S_n and such that there is an edge between vertices $\sigma, \tau \in S_n$ if and only if $\tau\sigma^{-1} \in T$.

Assuming that each edge of this graph has resistance 1 ohm, compute the resistance between the following two "opposite" permutations:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & 3 & \cdots & n-1 & n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix}.$$

5. Consider other groups G and their generators.

6. Suggest and study additional questions.

10. **Rich Sequences**

Let (a_i) be a sequence over a set S, that is the terms of the sequence are elements of S. An (i, n)-*order Hankel matrix* of the sequence is the following $n \times n$ matrix:

$$H_{i,n} = \begin{pmatrix} a_i & a_{i+1} & \dots & a_{i+n-1} \\ a_{i+1} & a_{i+2} & \dots & a_{i+n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i+n-1} & a_{i+n} & \dots & a_{i+2n-2} \end{pmatrix}.$$

An infinite sequence $(a_i)_{i\geq 1}$ is said to be *rich* if, for all positive integers i and n, the determinant $|H_{i,n}|$ is nonzero. A finite sequence $(a_i)_{1\leq i\leq N}$ is *rich* if the determinant $|H_{i,n}|$ is nonzero for all positive integers i and n such that $i + 2n - 2 \leq N$.

1. Let S be a subset of $\mathbb{Z}$ of cardinality 2.

a) Show that there exist no infinite rich sequences over S.
b) What is the maximum length of a rich sequence over S?
c) Describe all rich sequences when $S = \{1, 2\}$.

2. Let S be a subset of $\mathbb{Z}$ of cardinality 3.

a) Construct finite rich sequences when $S = \{1, 2, 3\}$.
b) Is there an infinite rich sequence over S?

3. Investigate the existence of infinite rich sequences when $S \subset \mathbb{Z}$ is a subset of cardinality more than 3.

4. Consider the infinite sequence $(a_i)_{i\geq 1}$ over the set $\{1, 2, 3, 4\}$ such that the infinite word $a_1a_2a_3\dots$ is invariant under the transformation $1 \mapsto 12$, $2 \mapsto 23$, $3 \mapsto 14$, $4 \mapsto 32$. The first sixteen terms of the sequence are 1, 2, 2, 3, 2, 3, 1, 4, 2, 3, 1, 4, 1, 2, 3, 2. Is the sequence $(a_i)_{i\geq 1}$ rich?

5. Investigate the existence of infinite rich sequences and construct finite rich sequences, when S is a subset of a ring $\mathbb{Z}_m$ of residues modulo m.

6. Investigate the existence of infinite rich sequences and construct finite rich sequences, when S is a subset of a finite field $\mathbb{F}_q$ of size $q = p^k$, where p is prime.

7. Suggest and study additional questions.

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