PROBLEMS FOR THE 7^{th} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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Contents

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Notation

$\mathbb{N} = \{1, 2, 3, \ldots\}$	set of natural numbers (positive integers)
$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$	ring of residue classes modulo n
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$	sets of integer, rational and complex numbers
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{F}_q$	finite field of size $q = p^k$
$GL(n, K)$	group of invertible $n \times n$ matrices over a field K
$\gcd(x_1, \ldots, x_n)$	greatest common divisor of $x_1, \ldots, x_n$
$\|x\|$	absolute value of x
$\#M$ or $\|M\|$	cardinality of a set M
$f(n) \ll g(n)$	one has $f(n) \leq g(n)$ for sufficiently large n

1. A Laser Machine

A metallic chip of shape P is put inside a unitary disk in the real plane. An engineer designed a machine that can move both ways around the disk on its circumference, can emit a laser beam of width w in any desired direction and can read whether or not the beam hit the chip.

The engineer wants to determine a characteristic σ of the chip with an error at most $\varepsilon > 0$.

1. Denote by $N(P)$ the minimal number of laser beams that the machine must emit in order to accomplish the engineer's goal. Find $N(P)$ or give an upper bound in the situations below (the cases A, B, C and D are independent of each other). Start with $\varepsilon = \frac{1}{3}, \frac{1}{4}, \frac{1}{10}$.

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A1) P is a circle;
A2) P is a convex n-gon;
A3) P is an arbitrary convex shape;
A4) P is not convex;

B1) P contains the center of the disk;
B2) P doesn't contain the center of the disk;

C1) σ is the diameter of P;
C2) σ is the perimeter;
C3) σ is the area;
C4) σ is another characteristic;

D1) w is negligible;
D2) w is not negligible.

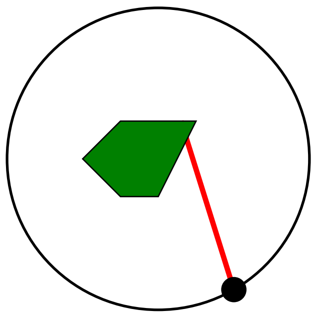

FIGURE 1. P is a convex pentagon and contains the center of the disk.

2. Now, instead of minimising $N(P)$, the engineer would like to minimise the total length $L(P)$ of the path that the machine must run on the circumference to determine σ with an error at most $\varepsilon > 0$. Find $L(P)$ or give an upper bound in the situations of the question 1.

3. Construct algorithms and write programs optimising $N(P)$ and $L(P)$ for a given P.

4. Suggest and study other directions of research.

2. **Maximal Minimal Triangles**

Let D be a convex domain in the plane with area 1. Let P_n be an arbitrary set of n distinct points inside or on the boundary of D. Finally, let $Min(P_n)$ be the minimal area of $n(n-1)(n-2)/6$ triangles formed by three of n points of P_n.

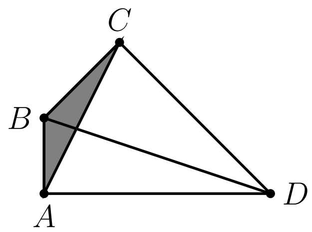

FIGURE 2. A set $P_4 = \{A, B, C, D\}$ with $Min(P_4) = area(ABC)$.

We are interested for the maximum of $Min(P_n)$, when D is fixed. Denote this number by $Max_{P_n} Min(P_n)$. In other words, $Max_{P_n} Min(P_n) = a$ if there exists a configuration P_n of n points inside D such that any triangle formed by 3 of these points has area at least a, and this is the maximal a for which we can say this. In fact, it is possible to prove that the maximum is always achieved.

1. Determine $Max_{P_n} Min(P_n)$ in the following cases:
a) D is a triangle (can be assumed to be equilateral) and $n = 4$;
b) D is a square and $n = 5$;
c) D is a triangle and $n = 5$.

2. Consider the case that D is bounded by a circle (or an ellipse), n is arbitrary.

3. Consider the case that D is a triangle and $n > 5$.

4. Consider other types of domains D.

5. One can also evaluate the expression $Min_D Max_{P_n} Min(P_n)$, where D belongs to a certain class of convex domains with area 1, for example, the set of all quadrilaterals. In other words, $Min_D Max_{P_n} Min(P_n) \geq a$, if for any D within a given class there exists a configuration of n points inside D, such that any triangle formed by 3 of these points has area greater than or equal to a.

6. Study the problem when D is a convex polygon and P_n contains all vertices of D plus a fixed number of inner points. One may start with the case that P_n is the set of vertices of D.

7. Suggest and investigate generalisations to three dimensions.

3. **Coloured Circles**

Let n and k be positive integers. Two high school students Clara and Carl play the following games.

1. In the beginning, Clara has n closed segments. She puts them one by one on a line in such a way that no point of the line is covered by more than k segments. Carl colours incoming segments so that any two intersecting segments have different colours (two segments are said to be intersecting if they have at least one point in common including the endpoints). Denote by $C_1(n,k)$ the minimum number of colours that Carl would need to successfully colour all n segments regardless of how Clara plays. Find $C_1(n,k)$ or give lower and upper bounds in the following cases:

a) the length of each segment is the same;
b) the segments are arbitrary.

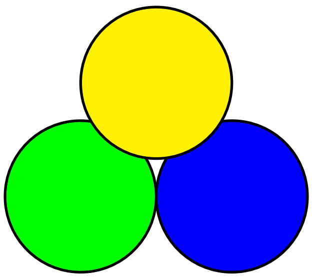

FIGURE 3. $C_2(3,2) = 3$.

2. Clara has n closed circles. She puts them one by one on a plane in such a way that no point of the plane is covered by more than k circles. Carl colours incoming circles so that any two intersecting circles have different colours (touching circles are also intersecting – at one point). Denote by $C_2(n,k)$ the minimum number of colours that Carl would need to successfully colour all n circles regardless of how Clara plays. Find $C_2(n,k)$ or give lower and upper bounds in the following cases:

a) the radius of each circle is the same;
b) the circles are arbitrary.

3. In the same manner, one can define a function $C_d(n,k)$ for a d-dimensional game. Determine relations between $C_d(n,k)$ when d varies. For example, what is $\lim_{d\to\infty} C_d(n,k)$?

4. Suggest and study other directions of research. For example, consider other types of figures.

4. **Simple Paths in Grids**

Consider an undirected graph G. Let V be the set of its vertices, and E the set of its edges. A *path of length* k is a sequence of distinct edges (v_0, v_1), (v_1, v_2), …, $(v_{k-1}, v_k) \in E$. We say that a path is *simple* if its vertices are distinct.

1. Let n and k be positive integers. Consider an $n \times n$ grid, which is a graph with $(n+1)^2$ vertices and $2n(n+1)$ edges. Denote by $D_n(k)$ the minimum number of edges that one should delete in this graph, so that the obtained graph wouldn't have a simple path of length k. For example, $D_n(1) = 2n(n+1)$. Find a formula for $D_n(k)$ or estimate it.

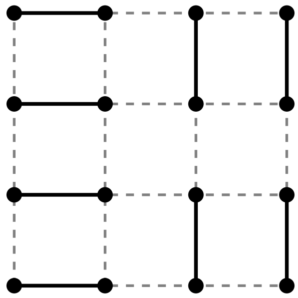

FIGURE 4. $D_3(2) \leq 16$ (the dashed edges are deleted).

2. Suppose that edges of the $n \times n$ grid are deleted randomly: for each edge, one tosses a coin to determine whether it should be deleted. Denote by $P_n(k)$ the probability that the obtained graph has no simple path of length k. Find $P_n(k)$ or estimate it.

3. Investigate a 3-dimensional generalisation.

4. Formulate and study an analogous problem for triangular grids.

5. Formulate and study an analogous problem for hexagonal grids.

6. Suggest other directions of research.

5. **A Recursive Sequence**

Let A be a subset of $[0,1]$. We take $u_0 \in [0,1]$ and define a sequence (u_n) by recursion in the following manner: u_{n+1} is the proportion of terms among $u_0, u_1, \ldots, u_n$ which belong to A. In other words, for $n \geq 0$ one has

$$u_{n+1} = \frac{\#\{i \mid 0 \leq i \leq n \text{ and } u_i \in A\}}{n+1}.$$

1. Does the sequence (u_n) converge? Study the following cases:

a) $A = [a, 1]$ where $0 < a < 1$;
b) $A = [0, a]$ where $0 < a < 1$;
c) $A = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k]$ with $k \in \mathbb{N}$ and $0 \leq a_1 < b_1 < \cdots < a_k < b_k \leq 1$;
d) A is arbitrary.

2. Denote $\ell_A(u_0) = \lim_{n\to\infty} u_n$ if the sequence converges, and let $\mathscr{L}_0(A)$ be the set of possible values $\ell_A(u_0)$ when u_0 varies in $[0,1]$. Find $\mathscr{L}_0(A)$ for the cases of the question 1.

3. Let N be a non-negative integer. Assume that one chooses real numbers $u_0, u_1, \ldots, u_N$ from the interval $[0,1]$, and constructs a sequence by using the recurrence relation above for $n \geq N$. Do the results of the question 1 still hold?

4. Let $\mathscr{L}_N(A)$ be the set of all possible limits $\lim_{n\to\infty} u_n$ in the question 3 and

$$\mathscr{L}_\infty(A) := \bigcup_{N\geq 0} \mathscr{L}_N(A).$$

For a given set B, does there exist A such that $\mathscr{L}_\infty(A) = B$? Consider the following cases:
a) B finite; b) $B = [0,1]$; c) $B = \mathbb{Q} \cap [0,1]$; d) B is an arbitrary subset of $[0,1]$.

5. Let A be now a subset of $[0,1]^2$. Study the convergence of sequences (u_n) such that

$$u_{n+1} = \frac{\#\{i \mid 0 \leq i \leq n \text{ and } (u_i, u_{n-i}) \in A\}}{n+1}.$$

Start with $A = [0,a] \times [0,b]$.

6. **Critical Points**

A *critical point* of a differentiable function f is any value x in its domain where $f'(x) = 0$. Consider the following function

$$F(x) = \sum_{i=1}^{n} \frac{c_i}{P_i(x)},$$

where n is a positive integer, c_i are nonzero real constants and $P_i(x)$ are distinct polynomials with real coefficients. The function F is defined everywhere in $\mathbb{R}$ except for the real zeros of the polynomials. Suppose that it is non-constant.

We would like to estimate the number $N(F)$ of distinct real critical points of F.

1. Suppose that the polynomials are linear, $P_i(x) = x + a_i$ where $a_i \in \mathbb{R}$. Describe the set of possible values of $N(F)$ when n is fixed and

a) the numbers a_i and c_i may vary;
b) the numbers a_i are fixed but the numbers c_i may vary;
c) the numbers c_i are fixed but the numbers a_i may vary.

2. Let $P_i(x) = x^2 + a_i x + b_i$ be quadratic polynomials with $a_i, b_i \in \mathbb{R}$ and $a_i^2 < 4b_i$ for all i form 1 to n.

a) Estimate $N(F)$. Is it true that $N(F) \leq 2n - 1$?
b) Find all possible values of $N(F)$ for a fixed n.

3. Investigate the problem for polynomials of higher degrees.

7. **Chain Stores**

Let f be a piecewise continuous function on an interval $[a,b]$ such that $\int_a^b f(x)dx = 1$. One can imagine that the interval is a street and the function f is the population density. The latter means that the number $\int_s^t f(x)dx \leq 1$, where $a \leq s \leq t \leq b$, gives the proportion of people living in the part $[s,t]$ of the street. Merchants place chain stores in the street.

1. Suppose that there is only one merchant who is building his n stores in such a way that the inhabitants of the street could shop as comfortably as possible. In other words, he is trying to minimise the following integral

$$\int_a^b dist(x,S) f(x)\, dx,$$

over subsets $S \subset [a,b]$ of n points, where S corresponds to the locations of the stores. Here $dist(x,M)$ denotes the distance of a point to the subset. What are the optimal configurations of the stores if

$$\text{a) } f(x) \equiv \frac{1}{b-a} \text{ ?} \qquad \text{b) } f(x) = -\frac{2}{b-a}\,|2x-(a+b)|+2 \text{ ?}$$

Start with $n = 1,2,3,4$.

2. Consider other functions f. Try to construct a general theory by finding regularities.

3. Let S_{opt} be an optimal set of n stores. Estimate the integral $\int_a^b dist(x,S_{opt})f(x)\,dx$.

4. Let us modify the problem. Suppose that the merchant is not able to build all his stores at once, and proceeds successively so that after each step, the inhabitants would find the locations of existing stores the most comfortable. How should the merchant place the stores? Start with the functions in 1a) and 1b). Compare the result with S_{opt} when n grows.

5. Suppose that there are two merchants M_1 and M_2. One after another, they successively build their stores in the street (first M_1 places a store, then M_2 places a store, then again M_1, etc.). Each merchant would like that, after each turn he plays, his stores be closer to as much people as possible and at the same time were comfortably located. More precisely, assume it is the turn of M_1 to build a store (the definitions are symmetric for M_2). Denote by $S_1 \subset [a,b]$ the locations of the stores of M_1 including the potential one, and $S_2 \subset [a,b]$ the locations of the stores of M_2 at the moment. Then M_1 wants to maximise the following integral

$$\int_{E_1} \left(l_1(x) - dist(x,S_1)\right) f(x)dx,$$

where $E_1 \subset [a,b]$ is the subset of points which are closer to S_1 than to S_2, that is

$$E_1 = \{x \in [a,b] \;\mid\; dist(x,S_1) > dist(x,S_2)\}\,.$$

The set E_1 being a union of intervals, $l_1(x)$ denotes the length of the interval which contains x.
Describe optimal configurations of stores.

6. Suppose that an inhabitant always goes to the closest store. Is it true that the ratio of the numbers of customers of the merchants tends to 1 as the total number of stores grows?

7. Suppose now that an inhabitant at point x always shops in a store of that merchant M_i for which the sum $\rho(x,S_i) = \sum_{y\in S_i} \frac{1}{(x-y)^2}$ is maximal. Each merchant M_i in his turn wants to maximise the integral $\int_{E_i} \left(l_i(x) - dist(x,M_i)\right) f(x)dx$, where

$$E_i = \{x \in [a,b] \;\mid\; \rho(x,M_i) > \rho(x,M_j),\; i \neq j\}\,,$$

and $l_i(x)$ is the length of the (maximal) interval from E_i which contains x. Investigate this situation.

8. Generalise and study the problem in the case where there are more than two merchants.

9. Extend the problem to a circle in the real plane, a ball or a sphere in $\mathbb{R}^3$. Construct optimal configurations for small values of n (total number of stores).

10. Construct a continuous time model, in which every merchant gets a profit proportionally to the number of customers. That profit is then invested in building new stores (the cost of a store and the time to build it are equal for everybody). Suppose that one merchant started his business well before all others. Will he be able to monopolise the region?

8. **A Diophantine Equation**

1. Denote by $f(n)$ the number of solutions of the equation

$$xyz + x + y = n$$

in positive integers x, y, z. Find all real $\varepsilon > 0$ such that $f(n) \ll n^{\varepsilon}$. The notation $f(n) \ll g(n)$ means that $f(n) \leq g(n)$ for sufficiently large n.

2. Let a and b be integers. Denote by $f(a, b)$ the number of solutions of the equation

$$xyz + a(x + y) = b$$

in positive integers x, y, z. Find all real $\varepsilon > 0$ such that

$$f(a, b) \leq |ab|^{\varepsilon}$$

whenever $|a|$ or $|b|$ is greater than some constant $N(\varepsilon)$. Estimate $N(\varepsilon)$.

3. Suggest and investigate other generalisations and directions of research.

9. **Framing Matrices**

Consider an n-dimensional vector space K^n over a field K. A collection F of vectors is said to be *in general position* if any n vectors from F are linearly independent over K. Note that by definition if $n > 1$ and F contains two copies of a vector, then it is not in general position.

Let A and B be $n \times n$ matrices with entries from K. The pair (A, B) will be called *framing* if there exists a vector $v \in K^n$ for which the collection

$$F = \left(A^k B^\ell \cdot v\right), \text{ where } 0 \leq k < \operatorname{ord}(A) \text{ and } 0 \leq \ell < \operatorname{ord}(B) \text{ are integers,}$$

is in general position. The *order* of a matrix, denoted by $\operatorname{ord}(C)$, is the least natural m such that C^m is the identity matrix (by convention $\operatorname{ord}(C) = +\infty$ if no such m exists). Note that the collection F is finite if and only if both matrices A and B are of finite order.

1. Let $K = \mathbb{C}$ and $\lambda = \exp\left(\frac{2\pi\sqrt{-1}}{n}\right)$. Consider the following matrices

$$A = \begin{pmatrix} \lambda^0 & & & 0 \\ & \lambda^1 & & \\ & & \ddots & \\ 0 & & & \lambda^{n-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ 1 & 0 & & 0 \end{pmatrix},$$

that is $A : (x_1, x_2, \dots, x_n) \mapsto (x_1, \lambda x_2, \dots, \lambda^{n-1} x_n)$ and $B : (x_1, x_2, \dots, x_n) \mapsto (x_2, \dots, x_n, x_1)$. The collection F constructed above consists of n^2 vectors.

a) Show that when n is prime, the pair (A, B) is framing.
b) Is there any n for which (A, B) is not framing?
c) Find all n for which (A, B) is framing.

2. Consider other pairs of matrices. In particular, investigate the following situations:

a) Both matrices are diagonal.
b) One of the matrices is not invertible.
c) One of the matrices is of infinite order.

3. Find framing pairs of matrices when $K = \mathbb{R}$.

4. Find framing pairs of matrices when K is a finite field. Start with $K = \mathbb{F}_p$ where p is prime.

10. **Polynomial Groups**

Let $\mathbb{F}_p$ be a finite field where p is prime. Consider the following set of polynomials over $\mathbb{F}_p$

$$G_n(p) = \left\{ x + a_2x^2 + \cdots + a_nx^n \ \mid \ a_i \in \mathbb{F}_p \right\}.$$

1. Show that $G_n(p)$ is a group under the operation of composition of functions modulo x^{n+1}. For example, the product of the elements $P = x + x^2$ and $Q = x + x^3$ in $G_4(p)$ is

$$P \circ Q = (x + x^3) + (x + x^3)^2 = x + x^2 + x^3 + 2x^4.$$

2. Find all pairs (n, p) for which the group $G_n(p)$ is abelian.

3. Is the group $G_n(p)$ two-generator?

4. Denote by $F(n, p)$ the number of generating pairs of the group $G_n(p)$. Estimate $F(n, p)$ by giving lower and upper bounds.

5. Let (P_1, P_2) be an arbitrary pair of elements in $G_n(p)$. Consider the following operations:

(A) interchange P_1 and P_2, that is $(P_1, P_2) \mapsto (P_2, P_1)$;

(B) replace P_i by P_i^{-1}, that is $(P_1, P_2) \mapsto (P_1^{-1}, P_2)$ or (P_1, P_2^{-1});

(C) replace P_i by $P_i \circ P_j$ or $P_j \circ P_i$, where $i \neq j$, for example $(P_1, P_2) \mapsto (P_1, P_1 \circ P_2)$.

Write $(P_1, P_2) \sim (Q_1, Q_2)$ if the pair (Q_1, Q_2) can be obtained from the pair (P_1, P_2) by applying any finite sequence of operations (A)–(C). All pairs of elements in $G_n(p)$ are thus divided into equivalence classes, denote by $N(n, p)$ the number of these classes.

Estimate $N(n, p)$. Start with $n = 2, 3, 4$.

6. Describe normal subgroups of $G_n(p)$.

7. Study other properties of groups $G_n(p)$.

E-mail address: oc@itym.org
URL: http://www.itym.org/