PROBLEMS FOR THE 6^{th} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

JULY 2014, BREMEN (GERMANY)

Contents

Keywords: 1. geometry, polygon – 2. additive number theory – 3. analysis, recursive sequence – 4. linear algebra, matrix norm – 5. Egyptian fraction, inequality – 6. analysis, continuous function – 7. Diophantine approximation, algebraic number – 8. permutation group, cycle – 9. number theory, residue class – 10. polynomial over a finite field, permutation – 11. combinatorial geometry.

Notation

$\mathbb{N} = \{1, 2, 3, \ldots\}$	set of natural numbers (positive integers)
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$	sets of integer, rational and complex numbers
$\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$	real line, real plane and 3-dimensional real space
$\gcd(x_1, \ldots, x_n)$	greatest common divisor of $x_1, \ldots, x_n$
$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$	ring of residue classes modulo n
$\mathbb{Z}_n^*$	group of invertible elements of $\mathbb{Z}_n$
$\mathbb{F}_q$	finite field of size $q = p^k$
$\mathbb{F}_q[X_1, \ldots, X_n]$	polynomial ring in symbols $X_1, \ldots, X_n$ over $\mathbb{F}_q$
S_n	symmetric group of degree n
$GL(n, \mathbb{R})$	group of invertible $n \times n$ matrices over $\mathbb{R}$
$M \times N$	direct product of two sets or two groups M and N
M^n	set of n-tuples of elements of a set M
$G \rtimes H$, $G\ wr\ H$	semidirect product and wreath product of groups G and H
$\|M\|$ or $\#M$	cardinality of a set M
$\|XY\|$	length of a segment XY
$\phi(n)$	Euler's totient function
$\log(x)$	natural logarithm of x
$\\|A\\|_p$	p-norm of a matrix A
$\limsup_{n\to+\infty} x_n$	limit superior of a sequence (x_n), $\limsup_{n\to+\infty} x_n = \lim_{n\to+\infty} \left(\sup_{m\geq n} x_m \right)$

Date: April 13, 2014.

1. **Poles of Polygons**

Let $n \geq 3$ and $k \geq 2$ be positive integers. Let $\mathcal{A} = A_1A_2 \ldots A_n$ be a convex polygon in the real plane $\mathbb{R}^2 \subseteq \mathbb{R}^k$. Denote by $\text{Pole}_k(\mathcal{A})$ the set of points $P \in \mathbb{R}^k$ for which it is possible to construct a **plane** polygon with side lengths $|PA_1|, |PA_2|, \ldots, |PA_n|$. (Here some lengths can be zero, the order of sides is arbitrary, non-convex polygons are accepted, sides must not have inner intersections.) The set $\text{Pole}_k(\mathcal{A})$ will be called the *pole* of $\mathcal{A}$ of dimension k.

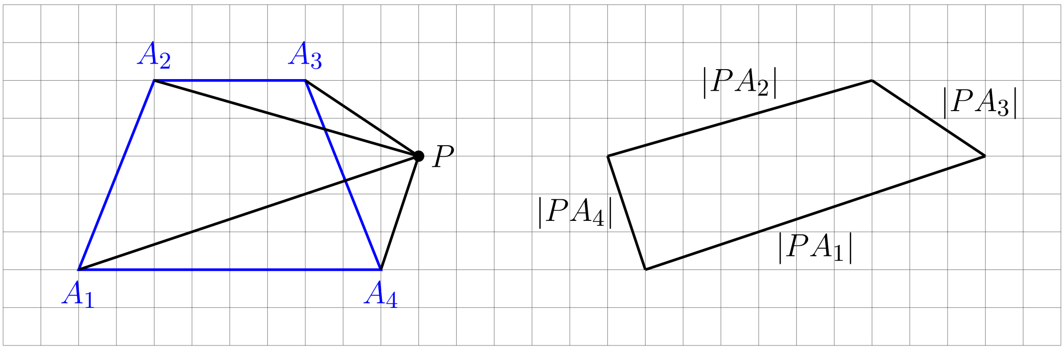

FIGURE 1. A point P belonging to the pole of a trapezoid $A_1A_2A_3A_4$.

1. Show that $\text{Pole}_2(\mathcal{A}) = \mathbb{R}^2$ for the following polygons:
a) an isosceles trapezoid, see Figure 1;
b) a regular polygon ($n \geq 4$);
c) a polygon with equal angles ($n \geq 4$).

2. Describe the poles of dimension 2 of convex n-gons. Start with small values of n.

3. Suppose that the vertices of $\mathcal{A}$ are integer points (*i.e.* their coordinates are integer). Take a point $P \in \text{Pole}_k(\mathcal{A})$ with integer coordinates. Does there exist a polygon with side lengths $|PA_1|, |PA_2|, \ldots, |PA_n|$ whose vertices are also integer points? First, consider $k = 2, 3$.

4. Denote by $\text{CPole}_k(\mathcal{A})$ the subset of points $P \in \mathbb{R}^k$ for which there exists a **plane convex** polygon with side lengths $|PA_1|, |PA_2|, \ldots, |PA_n|$. It will be called the *convex pole* of $\mathcal{A}$ of dimension k. Describe convex poles of dimension 2.

5. Given n and k, is it true that $\text{CPole}_k(\mathcal{A}) = \text{Pole}_k(\mathcal{A})$ for any convex n-gon $\mathcal{A}$?

6. Describe the poles and the convex poles of dimension 3. Start with small values of n.

7. Investigate the problem when $k \geq 4$.

2. **Free Subsets**

Let m, n and $k \leq m$ be positive integers. A subset $B \subseteq \{1, 2, \ldots, n\}$ will be called (k, m)-*free* if m cannot be presented as a sum of at most k distinct elements of B. For example, the set $\{1, 2, 3, 4, a \geq 5\}$ is $(3, 10)$-free if and only if $a \geq 11$. Denote by $f(k, m, n)$ the maximum cardinality of a (k, m)-free subset.

1. Study properties of $f(k, m, n)$ such as monotonicity, divisibility by a given prime, etc.

2. Give lower and upper bounds of the function. Start with small values of k.

3. Evaluate an asymptotic behaviour.

4. Find an exact formula.

5. Investigate the function $g(k, m, n)$ obtained by replacing "at most k distinct elements" in the statement of the problem with "exactly k distinct elements".

3. **A Recursive Sequence**

Let θ be a real number. Consider the sequence given by the following recurrent relation:

$$a_1 = 1 \quad \text{and} \quad a_n = 1 - \theta \cdot \sum_{k=1}^{n-1} \frac{a_k}{k \log(n+1-k)} \quad \text{for} \ \ n \geq 2.$$

1. If the limit of the sequence (a_n) exists, what is it equal to?

2. Show that $\lim_{n\to\infty} a_n$ exists when $0 \leq \theta < 1$. Consider also the case $\theta < 0$.

3. Investigate the case $\theta \geq 1$.

4. Study other recursive sequences by replacing the denominator in the relation above by the expression $k^r(\log(n+1-k))^s$, where r and s are positive real numbers.

4. **Binary Matrices**

Let n and $k \leq n$ be positive integers. Denote by $GL(n, \mathbb{R})$ the group of invertible $n \times n$ matrices with real entries. A matrix $A \in GL(n, \mathbb{R})$ will be called k-*binary* if it has the following property: in each column, exactly k entries are equal to 1 and all other entries are zeros.

Let p be a positive integer. The p-norm $\|x\|_p$ of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p},$$

and the p-norm $\|A\|_p$ of a matrix $A \in GL(n, \mathbb{R})$ is defined as

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

1. Calculate the number of k-binary matrices $A \in GL(n, \mathbb{R})$. What is the maximum determinant of a k-binary matrix of dimension n?

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Figure 2. A 2-binary matrix of dimension 4.

2. What are the minimum and maximum norms $\|A\|_p$ of a k-binary matrix of dimension n?

3. Find lower and upper bounds on the norm $\|A^{-1}\|_p$ of the inverse of a k-binary matrix of dimension n.

4. Given k and p, denote by $P_n(\tau)$ the probability that, for a k-binary matrix $A \in GL(n, \mathbb{R})$, one has the inequality $\|A^{-1}\|_p \leq n^\tau$. Does there exist a real number $\tau > 0$ such that

$$\limsup_{n\to+\infty} P_n(\tau) > 0\,?$$

5. Do there exist real constants $\alpha > 0$ and $\tau > 0$ depending on k and p such that

$$\|A^{-1}\|_p \leq \alpha \cdot n^\tau,$$

for any n and any k-binary matrix $A \in GL(n, \mathbb{R})$? Start with $k = 1, 2, 3, 4$ and $p = 1, 2$.

5. **An Egyptian Game**

Let n be a positive integer and let $E \subseteq \mathbb{N}$ be an infinite set of positive integers. Alice and Bob are playing the following game. First, Alice chooses n numbers $a_1 < a_2 < \ldots < a_n$ from the set E. After that Bob also chooses n numbers $b_1 < b_2 < \ldots < b_n$ from E (the same numbers may be chosen both by Alice and Bob). Bob wins if one of the next conditions is satisfied:

$$1 < \sum_{k=1}^{n} \frac{1}{b_k} < \sum_{k=1}^{n} \frac{1}{a_k} \qquad \text{or} \qquad \sum_{k=1}^{n} \frac{1}{a_k} \leq 1.$$

Otherwise, it is Alice who wins. Denote by $G(E) \subseteq \mathbb{N}$ the set of those positive integers n for which Alice has a winning strategy.

1. Find $G(E)$ in the following cases:
- a) $E = \{2^k \mid k \in \mathbb{N} \cup \{0\}\}$;
- b) $E = \mathbb{N}$;
- c) $E = 3 \cdot \mathbb{N}$;
- d) $E = \{p \mid p \text{ is prime}\}$.

2. Which subsets of $\mathbb{N}$ can be presented as $G(E)$ for some E.

3. Is there a sequence of infinite sets $(E_i)_{i \in \mathbb{N}}$ such that $E_{i+1} = G(E_i)$, for all $i \in \mathbb{N}$?

4. Alice and Bob decided to change the conditions: Bob wins if and only if

$$1 > \sum_{k=1}^{n} \frac{1}{b_k} > \sum_{k=1}^{n} \frac{1}{a_k} \qquad \text{or} \qquad \sum_{k=1}^{n} \frac{1}{a_k} \geq 1.$$

Show that for all n, Alice has a winning strategy.

5. We say that Alice has a *lazy winning strategy* in the latter game if there exists an infinite subset $L \subseteq E$ such that, by choosing the n smallest numbers of L, Alice wins for all $n \in \mathbb{N}$. Determine whether Alice has a lazy winning strategy, when E is as in 1a) - d).

6. Suggest and study additional directions of research.

6. **Ridges of Functions**

Let X be a set and $f : X \to X$ be a function. The *preimage* $f^{-1}(x)$ of a point $x \in X$ is the set of all points $y \in X$ such that $f(y) = x$. The *ridge* of the function f is the set of cardinalities of the preimages $f^{-1}(x)$,

$$ridge(f) = \left\{ \# f^{-1}(x) \;\mid\; x \in X \right\}.$$

If the preimage of x is infinite, then we put $\# f^{-1}(x) = \infty$.

1. Take $X = \mathbb{R}$. We are interested in the ridges of continuous functions $f : \mathbb{R} \to \mathbb{R}$.
- a) Find all integers n such that $ridge(f) = \{n\}$ for a continuous function f.
- b) Can $ridge(f)$ contain only even integers?
- c) Describe subsets of $\mathbb{N} \cup \{0, \infty\}$ which are ridges of polynomials.
- d) Describe subsets of $\mathbb{N} \cup \{0, \infty\}$ which are ridges of continuous functions.

2. The same questions when $X = [0, 1]$.

3. An analogous problem for $X = \mathbb{R}^2$.

4. Consider other sets X.

7. **Decimal Representations**

Let $\alpha = (a_n)_{n\in\mathbb{N}}$ be a sequence of positive integers. We denote by $D(\alpha)$ the real number $0, a_1a_2...a_n...$ with the decimal representation obtained by writing successively all the terms of the sequence α. For example, $D(\alpha) = 0,1234567891011...$ when α is the sequence of successive positive integers.

1. Determine whether the number $D(\alpha)$ is rational or irrational, if α is

a) the sequence of successive positive integers: $1, 2, 3, 4, 5, \ldots$;
b) the sequence of successive prime numbers: $2, 3, 5, 7, 11, \ldots$;
c) the Fibonacci sequence: $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$.

2. Find sufficient and/or necessary conditions on α for the number $D(\alpha)$ to be rational.

3. In the examples 1a) - c) above, is $D(\alpha)$ algebraic or transcendental?

4. Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of the set of positive integers, *i.e.* f is a bijection. Denote by α_f the sequence $(f(n))_{n\in\mathbb{N}}$. Determine permutations f for which the number $D(\alpha_f)$ is

a) rational;
b) algebraic;
c) transcendental.

5. Consider other sequences α, and determine whether $D(\alpha)$ is algebraic or transcendental.

6. Find sufficient and/or necessary conditions on α for the number $D(\alpha)$ to be algebraic.

8. **Cycles in a Permutation Group**

Denote by S_n the group of permutations of the set $\{1, 2, \ldots, n\}$. Let G be a subgroup of S_n. Introduce the set

$$C(G) = \{s \in G \mid s \text{ is a cycle of length } n\}.$$

We would like to know whether the following inequality is correct:

$$\frac{|C(G)|}{|G|} \leq \frac{\phi(n)}{n}, \tag{1}$$

where $\phi(n)$ is the Euler's totient function.

1. Show that if $G = \langle s \rangle$ is a cyclic group generated by a cycle s of length n, then one has the equality in (1).

2. Check the inequality when G is a direct product $C_a \times C_b$, a semidirect product $C_a \rtimes C_b$ or a wreath product $C_a \ wr \ C_b$ of two cyclic groups of orders a and b.

3. Describe all permutation subgroups for which the equality holds.

4. Check the inequality for primitive permutation subgroups.

5. Is the inequality correct for any subgroup of S_n?

6. More generally, let $2 \leq k \leq n$ be a positive integer. Consider the following set:

$$C_k(G) = \{s \in G \mid s \text{ is a cycle of length } k\}.$$

Suggest an upper bound for the density $|C_k(G)|/|G|$ depending on k and n. Adapt and answer the previous questions. You may start with small values of k and n.

9. **An Equation in $\mathbb{Z}_n$**

Consider the ring $\mathbb{Z}_n$ of residues modulo a positive integer n. Denote by $\mathbb{Z}_n^*$ the subset of invertible elements, one has $\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n \mid \gcd(x,n) = 1\}$.

Let p be an odd prime number. We want to study the equation

$$x + x^{-1} + y + y^{-1} = t\,,$$

where $x, y \in \mathbb{Z}_p^*$ and $t \in \mathbb{Z}_p$. Denote by $N_p(t)$ the number of its solutions (x,y) for a given t.

1. Let $A = \{x + x^{-1} \mid x \in \mathbb{Z}_p^*\}$. What are the cardinalities of A and the sets

$$A + A = \{a + b \mid a, b \in A\} \quad \text{and} \quad A \cdot A = \{a \cdot b \mid a, b \in A\}?$$

2. Find $N_p(0)$. For small values of p, give an explicit formula of $N_p(t)$ depending on t.

3. Describe maps $f : t \mapsto \frac{at+b}{ct+d}$ under which N_p is invariant, that is, $N_p(t) = N_p(f(t))$:

a) Find all $a \in \mathbb{Z}_p^*$ and $b \in \mathbb{Z}_p$ such that $N_p(t) = N_p(at + b)$ for any $t \in \mathbb{Z}_p$.

b) Determine $b \in \mathbb{Z}_p^*$ such that $N_p(t) = N_p(bt^{-1})$ for any $t \in \mathbb{Z}_p^*$.

c) Consider other maps f.

4. Study an analogous problem for the ring $\mathbb{Z}_{p^k}$ where p is a prime and $k \in \mathbb{N}$.

5. Generalize results to $\mathbb{Z}_n$ for any integer n.

10. **Reflexive Polynomials**

Let n and k be positive integers, p be a prime number and $q = p^k$. Denote by $\mathbb{F}_q$ a finite field of size q, and let $\mathcal{R} = \mathbb{F}_q[X_1, \ldots, X_n]$ be the ring of polynomials in symbols $X_1, \ldots, X_n$ over the field $\mathbb{F}_q$.

1. Is it true that for any permutation $\sigma : \mathbb{F}_q^n \to \mathbb{F}_q^n$ of the set $\mathbb{F}_q^n$, there exists an n-tuple of polynomials $(P_1, \ldots, P_n) \in \mathcal{R}^n$ such that

$$(P_1(x_1, \ldots, x_n), \ldots, P_n(x_1, \ldots, x_n)) = \sigma(x_1, \ldots, x_n), \quad \text{for all } x_1, \ldots, x_n \in \mathbb{F}_q\,? \qquad (2)$$

Start with $n = 1, 2$ and $k = 1, 2$.

An element $P = (P_1, \ldots, P_n) \in \mathcal{R}^n$ is called a *reflexive map* if there exists an element $Q = (Q_1, \ldots, Q_n) \in \mathcal{R}^n$ such that the composition $P \circ Q$ is the identity map $(X_1, \ldots, X_n)$, that is,

$$(P_1(Q_1, \ldots, Q_n), \ldots, P_n(Q_1, \ldots, Q_n)) = (X_1, \ldots, X_n).$$

For instance $P = (X + Y^2, Y) \in (\mathbb{F}_q[X,Y])^2$ is a reflexive map, its inverse is $Q = (X - Y^2, Y)$.

2. Suppose that p is odd. Find all permutations σ of the set $\mathbb{F}_q^n$ which are induced by a reflexive map $P = (P_1, \ldots, P_n) \in \mathcal{R}^n$, *i.e.* one has the equality (2). Start with $n = 1, 2$ and $k = 1, 2$.

Note that for example the element $(X^q, Y) \in (\mathbb{F}_q[X,Y])^2$ is not a reflexive map, even though it induces a bijection of $\mathbb{F}_q^2$.

3. The same problem when $p = 2$. In particular, does there exist an **odd** permutation of $\mathbb{F}_q^n$ that is induced by a reflexive map?

11. **Points, Lines and Planes**

Let N points be given in the space $\mathbb{R}^3$, not all on the same plane and also not all lying on two lines. Denote by L the number of lines passing through at least two of these points, and let P be the number of planes containing at least three of the points.

1. For small N, what values can the expression $N - L + P$ take?

2. Is it true that $N - L + P \geq 0$ for sufficiently large N?

3. Suppose that no three points are collinear. Is it true that $N - L + P \geq 2$ for any sufficiently large N? When does the equality hold?

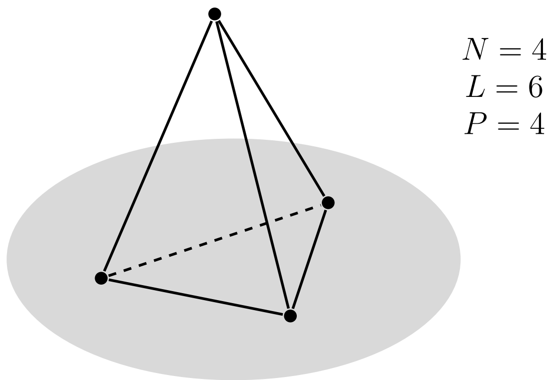

FIGURE 3. A configuration of points with $N - L + P = 2$.

4. Determine the set of possible values of the expression $N - L + P$ for a given N. Distinguish the general situation and the case that no three points are collinear.

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