PROBLEMS FOR THE 4^{th} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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Contents

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Notation

$\mathbb{N} = \{1, 2, 3, \ldots\}$	set of natural numbers (positive integers)
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$	sets of integer, rational and complex numbers
$\mathbb{R}$, $\mathbb{R}^2$	real line and real plane
$a \bmod b$	remainder (residue) after the division of a by b
$\mathbb{F}_p = \{0, 1, \ldots, p-1\}$	field of residue classes modulo a prime p
$\#M$	cardinal of a set M
$\tau(n)$	number of natural divisors of n
$\varphi(n)$	Euler's totient function
$\gcd(x_1, \ldots, x_n)$	greatest common divisor of $x_1, \ldots, x_n$
$\limsup\limits_{n\to+\infty} x_n$	limit superior of a sequence (x_n), $\limsup\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} \left(\sup\limits_{m\geq n} x_m\right)$

1. Generalizing Perfectness

Any function $f\colon \mathbb{N} \to \mathbb{C}$ is called *arithmetic.* Recall that a natural number $n \geq 1$ is said to be *perfect* if it is equal to the sum of its positive proper divisors (that is all divisors except n). Examples of perfect numbers are 6, 28 and 496. Euclid proved that if k is a natural number such that $2^k - 1$ is prime, then $n = 2^k(2^k - 1)$ is perfect.

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Generalizing this notion, we say that a natural number $n \in \mathbb{N}$ is *f-perfect* for some arithmetic function f if

$$f(n) = \sum_{\substack{d|n \\ 1 \le d \le n-1}} f(d).$$

Thus, n is perfect if and only if n is f-perfect for $f(n) = n$. For another example, note that a natural number $n \ge 1$ is f-perfect for the constant function $f(n) = 1$ if and only if n is a prime.

1. Let $\tau(n)$ denote the number of positive divisors of n (including n).

a) Prove that a natural number $n \ge 1$ is τ-perfect if and only if n is the square of a prime.

b) Find all f-perfect natural numbers $n \ge 1$ for the function $f(n) = \tau(n) - 1$. For as many values of $k \in \mathbb{Z}$ as possible, find all f-perfect natural numbers $n \ge 1$ for $f(n) = \tau(n) + k$.

2. Find all f-perfect numbers n, where $f(n) = \varphi(n)$ is Euler's totient function.

3. a) Prove that if k is a natural number such that $2^{k+1} - 2k - 1$ is a prime, then $n = 2^k(2^{k+1} - 2k - 1)$ is f-perfect for $f(n) = n - 1$.

b) Find similar sufficient conditions for f-perfectness for other polynomial functions of degree 1 such as $f(n) = n - 2$ or $f(n) = n + 1$.

4. Let $f(n) = \ln(n)$. Find all f-perfect numbers n.

5. Let $f(n) = (-1)^n$. Find all f-perfect numbers n. Study the general case that $f(n) = \omega^n$, where $\omega \in \mathbb{C}$ is a root of unity.

6. Let $f(n) = \binom{2012}{n}$. Find all f-perfect numbers n. Study the general case where 2012 is replaced by a natural number m.

7. Consider other arithmetic functions f and find sufficient and/or necessary conditions for a number to be f-perfect.

8. Recall that a pair (m, n) of positive integers is said to be *amicable* if

$$n = \sum_{\substack{d|m \\ 1 \le d \le m-1}} d \quad \text{and} \quad m = \sum_{\substack{d|n \\ 1 \le d \le n-1}} d.$$

An example is $(220, 284)$. For an arithmetic function f, give a reasonable definition for a pair (m, n) of positive integers to be f-amicable. For various arithmetic functions f, find f-amicable pairs or prove that no f-amicable pair exists.

2. **A Number of Residues**

Denote by $T_k = \frac{k(k+1)}{2}$ the k^{th} triangular number, where $k \in \mathbb{N}$. For each positive integer n, let u_n be the number of distinct members of the sequence $(T_k \mod n)_{k \ge 1}$.

1. Find a formula for u_n when n is a power of 2.

2. Find a formula for u_n when n is a power of a prime number.

3. Find a formula for u_n in general case.

4. Let $P(x)$ be a polynomial with rational coefficients such that $P(k)$ is integer for every $k \in \mathbb{Z}$. Find the number of distinct residues modulo n in the sequence $(P(k))_{k \ge 1}$, where n is a positive integer.

5. Suggest and investigate related questions.

3. **Sums of Powers**

Let $n \geq 2$ and let $z_1, z_2, ..., z_n$ be complex numbers. Define $S_k = z_1^k + z_2^k + ... + z_n^k$. We assume that S_k is an integer for all $k \geq 1$.

1. a) Prove that the polynomial $(X - z_1) \cdot (X - z_2) \cdot ... \cdot (X - z_n)$ has integer coefficients.

b) If $z_1, z_2, ..., z_n$ are rational numbers, prove that they are all integers.

2. a) Suppose that $z_1, z_2, ..., z_n$ are integers. Give a necessary and sufficient condition for the following to be true: there are infinitely many primes p dividing at least one of the numbers $S_1, S_2, S_3, ...$

b) Answer the same question in the general case.

3. a) Find $z_1, z_2, ..., z_n$ such that S_k is the k^{th} power of a rational number for all $k \geq 1$. You may start with the case that $z_1, z_2, ..., z_n$ are all real numbers.

b) Find $z_1, z_2, ..., z_n$ such that $\frac{1}{n} S_k$ is the k^{th} power of a rational number for all $k \geq 1$.

4. Let a_k be the first digit of the integer S_k and assume that $(a_k)_k$ is a periodic sequence. What can be said about the numbers $z_1, z_2, ..., z_n$? First, you may consider the case that all z_i are integers.

5. Investigate the same questions for the sequence $S_k = \sum_{i=1}^{n} f_i(n) z_i^n$, where f_i are polynomials with complex coefficients.

4. **Isosceles Triangles**

Consider a triangle ΔABC with side lengths a, b, c and median lengths m_a, m_b, m_c as shown in Figure 1. A well-known fact is that $m_a = m_b$ if and only if $a = b$, *i.e.* the triangle is isosceles.

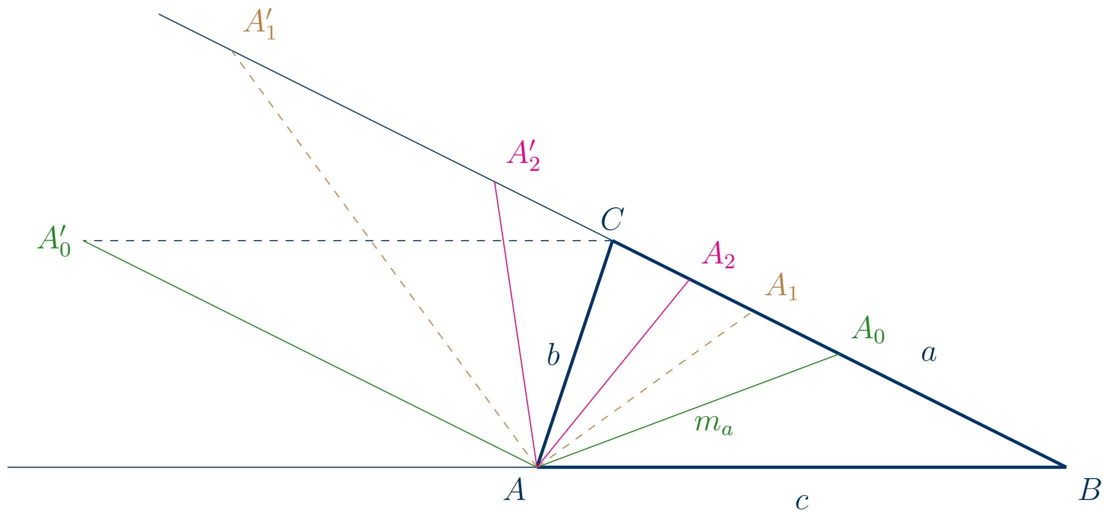

FIGURE 1. ΔABC with a median AA_0, an internal bisector AA_1, a symmedian AA_2, an exmedian AA'_0, an external bisector AA'_1 and an exsymmedian AA'_2.

1. Prove that two internal angle bisectors of a triangle are equal if and only if the triangle is isosceles.

2. The *symmedian* through a given vertex of a triangle is constructed by reflecting the median about the internal angle bisector at the same vertex. Prove that two symmedians of a triangle are equal if and only if the triangle is isosceles.

3. Is it true that two external angle bisectors of a triangle are equal if and only if the triangle is isosceles?

4. An *exmedian* is a parallel to a side of a triangle through the opposite vertex. The *exsymmedian* through a vertex of a triangle is constructed by reflecting the exmedian about the external angle bisector at the same vertex. Is it true that two exsymmedians of a triangle are equal if and only if the triangle is isosceles?

Let n be a nonzero real number. An *internal* (*external*) n-*line* of the triangle is a line through the vertex that divides internally (externally) the opposite side in proportion of the n^{th} powers of the adjacent sides. Namely, segments AA_n and AA'_n are internal and external n-lines respectively at the vertex A if one has $\frac{BA_n}{A_nC} = \frac{c^n}{b^n}$ and $\frac{BA'_n}{A'_nC} = \frac{c^n}{b^n}$.

5. Check that the internal bisectors and the symmedians are respectively internal 1-lines and 2-lines of the triangle. Also, the external bisectors and the exsymmedians are respectively external 1-lines and 2-lines of the triangle.

6. Is it true that two internal n-lines of a triangle are equal if and only if the triangle is isosceles?

7. Is it true that two external n-lines of a triangle are equal if and only if the triangle is isosceles?

8. Suggest and study additional directions of research.

5. **Stable Polygons**

Let $n \geq 3$ be a positive integer, and let P_n be the set of vertices of a regular n-gon. A subset $A \subseteq P_n$ is called *stable* if the center of gravity of the points in A coincides with the center of the regular n-gon.

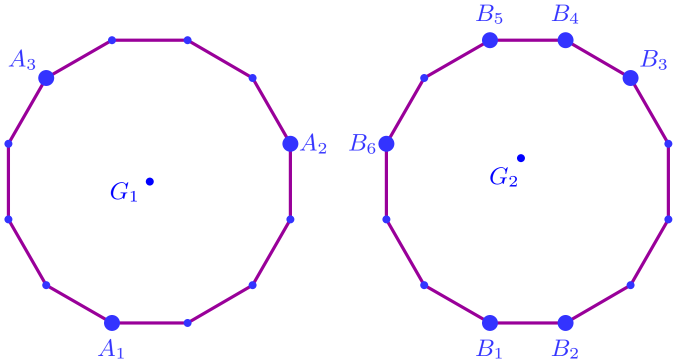

FIGURE 2. The subset $A = \{A_1, A_2, A_3\}$ of P_{12} is stable, but the subset $B = \{B_1, \ldots, B_6\}$ is not.

1. When n is prime, find the number of stable subsets $A \subseteq P_n$ and describe them.

2. The same problem when n is the product of two distinct prime numbers.

3. The same problem when n is a power of a prime number.

4. Investigate the problem for an arbitrary n.

5. Suggest and study additional directions of research.

6. **Recurrent Sequences**

1. Let $(u_n)_{n\geq 1}$ be a sequence of real numbers such that, for any $n \geq 2$,

$$u_{n+1} = \frac{u_1^2 + u_2^2 + \cdots + u_n^2}{n}.$$

Study properties of the sequence $(u_n)_{n\geq 1}$ depending on u_1 and u_2. For example, is it monotonic, bounded, convergent, ...?

2. Let $(u_n)_{n\geq 1}$ be a sequence of real numbers such that, for any $n \geq 2$,

$$u_{n+1} = \frac{u_1 u_n + u_2 u_{n-1} + \cdots + u_{n-1} u_2 + u_n u_1}{n}.$$

Study properties of the sequence $(u_n)_{n\geq 1}$ depending on u_1 and u_2.

3. Let u_1 and u_2 be real numbers. A random sequence $(u_n)_{n\geq 1}$ is constructed as follows. For each $n \geq 2$, if $u_1, u_2, \ldots, u_n$ are already constructed, choose a random permutation σ of the set $\{1, 2, \ldots, n\}$ with probability $1/n!$ and set

$$u_{n+1} = \frac{u_1 u_{\sigma(1)} + u_2 u_{\sigma(2)} + \cdots + u_n u_{\sigma(n)}}{n}.$$

a) Study properties of the sequence $(u_n)_{n\geq 1}$ depending on u_1 and u_2.
b) Same question if σ is a random function $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ instead of a random permutation.

4. Let $(u_n)_{n\geq 1}$ be a sequence of real numbers such that, for any $n \geq 4$,

$$u_n = \frac{u_1 u_{n+1} + u_2 u_n + \cdots + u_n u_2 + u_{n+1} u_1}{n+1}.$$

a) Study properties of the sequence $(u_n)_{n\geq 1}$ depending on u_1, u_2, u_3 and u_4.
b) Suppose in addition that $u_1 = u_2 = u_3 = 1$. Does there exist a value of u_4 such that $0 \leq u_4 < 1$ and $u_n < 1$ for every $n \geq 1$?

7. **An Experiment**

Consider the set $\mathbb{F}_2 = \{0, 1\}$ of residues modulo 2 with addition rules $0 + 0 = 0$, $0 + 1 = 1$ and $1 + 1 = 0$. Let n be a positive integer. Define an operation α on the set of n-tuples:

$$\alpha(x_1, x_2, \ldots, x_n) = (x_1 + x_2, x_2 + x_3, \ldots, x_n + x_1), \quad \text{for any} \quad (x_1, x_2, \ldots, x_n) \in (\mathbb{F}_2)^n.$$

We say that an n-tuple is *balanced* if exactly half of its elements are zeros (in this case n is even). A balanced tuple $X = (x_1, x_2, \ldots, x_n) \in (\mathbb{F}_2)^n$ is called *superbalanced* if its images

$$\alpha(X), \quad \alpha(\alpha(X)), \quad \ldots, \quad \alpha^k(X) = \underbrace{\alpha(\alpha(\ldots\alpha(}_{k} X))), \quad \ldots$$

are balanced for all $k \in \mathbb{N}$.

1. a) Let ℓ be a positive integer. Find the number of balanced n-tuples $X = (x_1, x_2, \ldots, x_n)$ in the set $(\mathbb{F}_2)^n$ such that the n-tuples $\alpha(X), \ldots, \alpha^\ell(X)$ are balanced as well. Start with $\ell = 1, 2, 3$.

b) Do there exist superbalanced n-tuples? How many?

2. Invent and investigate your own problem analogous or related to the given one (even if you have no answer to the first question). As an idea, you may generalize the problem and consider other cases, or slightly reformulate it by modifing the conditions. A problem suggested by your team must be new (unsolved in literature, in papers and over the internet).

8. **Discharged Batteries**

1. A racer has an electric car that needs two batteries to run. Unfortunately, the only reserve at his disposal is a set of 10 batteries, and the only thing he knows is that five of them are charged and the other five are uncharged. Call a *2-try* the act of putting two batteries in the car and trying to make it run. What strategy would you suggest to the racer to apply in order to minimize the number of 2-tries needed to find a charged pair?

2. Now, suppose that the car is bigger and it needs three charged batteries instead. Suppose also that the racer has only eight batteries, but still a symmetric distribution between charged and uncharged. Give a strategy to minimize the number of 3-tries.

3. Investigate the case where there are $c > 2$ charged batteries and $u > 0$ uncharged ones. What is the order of growth of the number of 2-tries as a function of c and u? Is there anything special about the case $c = u$?

4. Generalize the problem for triples (c, u, n), where $c > 2$ is the number of charged batteries, $u > 0$ is the number of uncharged ones, and the car needs $n < c$ batteries to run. Find a strategy minimizing the number of n-tries to make the car work.

5. A further generalization: given positive integers $c, u, n \geq 2$ and $m > n$, what is the best strategy to find m charged batteries among c charged and u uncharged batteries by using an "n-testing-car"?

6. Investigate the case where the car could run with half-charged batteries, but not all being half-charged. We suppose that (contrary to most real-world batteries) a half-charged battery furnishes half of the power of a fully charged one. Consider a tuple (c, h, u, n, p) of positive integers, where c is the number of charged batteries, h is the number of half-charged batteries, u is the number of uncharged ones, and the car needs n batteries with a total power $p \leq n$. Find a strategy minimizing the number of n-tries to make the car work.

7. Suppose that in the previous situation you don't want to run the car, but rather find $k \leq c$ among the charged batteries. How does the strategy change?

9. **Random Games**

Given a graph G and a vertex v, consider the following two-player games. Both games begin with a token placed on v, and the players alternately move the token from its current vertex to a neighbouring one, never returning to already visited vertices. In the game $\mathcal{G}_1$, a player unable to move the token in this way **loses**. In the game $\mathcal{G}_2$, a player who is unable to move **wins**. The games may be a win for the first player, a win for the second one or else a draw (if the graph is infinite). The goal of this problem is to play such games on random graphs.

Let $\mu = (\mu_i)_{i \geq 0}$ be a sequence of nonnegative real numbers such that

$$\sum_{i=0}^{\infty} \mu_i = 1 \quad \text{and} \quad \mu_0 + \mu_1 < 1.$$

The *Galton-Watson tree* with offspring distribution μ, or shortly the GW_μ *tree*, is a random plane tree defined as follows. Start with a single vertex (the *ancestor*) having a random number of children distributed according to μ, which means that the probability of having i children is μ_i. Then each of its children has itself an independent random number of children distributed according to μ, and so on. The genealogical tree of this population is the GW_μ tree. It is a random finite or infinite tree. We denote by p_{ext} the probability that the GW_μ tree is finite.

1. For $0 \leq x \leq 1$, consider the series

$$F(x) = \sum_{i=0}^{\infty} \mu_i x^i.$$

a) Show that p_{ext} is the smallest solution of the equation $F(x) = x$ in the interval $[0, 1]$.
b) Give a necessary and sufficient condition in order to have $p_{\text{ext}} = 1$.

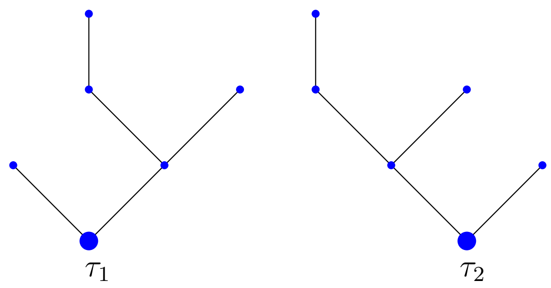

FIGURE 3. Two distinct plane trees τ_1 and τ_2.

Two players play the games $\mathcal{G}_1$ and $\mathcal{G}_2$ on the GW_μ tree starting from the ancestor. Set:

$$w = \mathbb{P}\,(\text{first player wins the game } \mathcal{G}_1)\,, \qquad \widetilde{w} = \mathbb{P}\,(\text{first player wins the game } \mathcal{G}_2)$$
$$l = \mathbb{P}\,(\text{first player loses the game } \mathcal{G}_1)\,, \qquad \widetilde{l} = \mathbb{P}\,(\text{first player loses the game } \mathcal{G}_2)$$
$$d = \mathbb{P}\,(\text{draw in the game } \mathcal{G}_1)\,, \qquad \widetilde{d} = \mathbb{P}\,(\text{draw in the game } \mathcal{G}_2)$$

2. Estimate $w, l, d, \widetilde{w}, \widetilde{l}, \widetilde{d}$.

3. Find inequalities between $w, l, d, \widetilde{w}, \widetilde{l}, \widetilde{d}$.

4. For $\lambda > 0$ and $i \geq 0$, set:

$$\mu_i^{(\lambda)} = \frac{\lambda^i}{i!} e^{-\lambda}.$$

Let $w^{(\lambda)}, l^{(\lambda)}, d^{(\lambda)}, \widetilde{w}^{(\lambda)}, \widetilde{l}^{(\lambda)}, \widetilde{d}^{(\lambda)}$ be defined as previously. Are these functions continuous with respect to λ? Differentiable with respect to λ?

5. Investigate games $\mathcal{G}_1$ and $\mathcal{G}_2$ on other types of random graphs.

10. **Densities of Natural Subsets**

Introduce the following functionals on the set of all subsets of $\mathbb{N}$:

$$\mu_1(E) = \limsup_{n \to +\infty} \frac{\#(E \cap [1, n])}{n}, \quad \text{for any } E \subseteq \mathbb{N},$$

$$\mu_2(E) = \limsup_{x \to 1^-} (1 - x) \sum_{n \in E} x^n, \quad \text{for any } E \subseteq \mathbb{N},$$

$$\mu_3(E) = \limsup_{x \to 1^+} (x - 1) \sum_{n \in E} \frac{1}{n^x}, \quad \text{for any } E \subseteq \mathbb{N}.$$

Real numbers $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ will be called *densities* of a set E.

1. Show that the densities μ_1, μ_2 and μ_3 are well-defined.

2. What are the densities $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ of a finite set E?

3. Find the densities $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ of an arithmetic progression

$$E = \{a + nd \ \mid \ n \in \mathbb{N} \cup \{0\}\}\,,$$

where a and d are positive integers.

4. Find the densities $\mu_1(E)$, $\mu_2(E)$ and $\mu_3(E)$ of a set

$$E = \{[x^n] \ \mid \ n \in \mathbb{N} \cup \{0\}\}\,,$$

where $x \geq 1$ is a real number and $[\cdot]$ stands for the integral part.

5. Find or estimate the densities of other subsets of $\mathbb{N}$, for instance, of the Fibonacci sequence. Try to construct subsets with "interesting" densities.

6. Is it true that the following equality

$$\mu_1(A \cup B) + \mu_1(A \cap B) = \mu_1(A) + \mu_1(B)$$

holds for any subsets $A, B \subseteq \mathbb{N}$?

7. The same question for μ_2 and μ_3.

8. Introduce and investigate other densities on natural subsets.

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