PROBLEMS FOR THE 2^{nd} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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Contents

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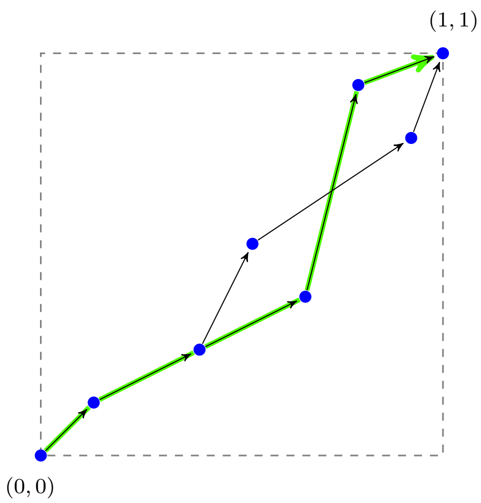
1. Blocking Sets

Let $S = \{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\}$ be a set of n points in the square $[0,1] \times [0,1]$ such that $a_i \neq a_j$ and $b_i \neq b_j$ for any $i \neq j$. Suppose that S contains the points $(0,0)$ and $(1,1)$.



Call such a set *acceptable*.
We draw an arrow from a point $(a_i, b_i) \in S$ to another point $(a_j, b_j) \in S$ if the following conditions are satisfied:

i) $a_i < a_j$ and $b_i < b_j$,

ii) either there is no point $(a_k, b_k) \in S$ such that $a_i < a_k < a_j$, or there is no point $(a_k, b_k) \in S$ such that $b_i < b_k < b_j$.

This construction gives a graph that we denote by G_S. Remark that in the obtained graph there are at most two arrows starting at a vertex.
A directed path in G_S (that is, a path following the arrows) starting at $(0,0)$ and ending at $(1,1)$ is called *diagonal* (see the figure).

We say that an acceptable set S is *blocking* if the graph G_S has no diagonal path.

1. What is the minimum number of elements that a blocking set can have?

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2. Two acceptable sets $S = \{(a_i, b_i)\}$ and $S' = \{(a'_i, b'_i)\}$ of n points are called *equivalent*, we write $S \sim S'$, if $b_1 < b_2 < \cdots < b_n$, $b'_1 < b'_2 < \cdots < b'_n$ and for any $i \neq j$ we have $a_i < a_j$ if and only if $a'_i < a'_j$. In particular, the graphs of equivalent sets are isomorphic. Denote by p_n the probability of choosing a blocking set, that is, the ratio of the number of non-equivalent blocking sets of n points to the number of all non-equivalent acceptable sets of n points. Calculate or estimate p_n. What is the limit of this probability as $n \to \infty$?

3. Let $D(S)$ be the number of diagonal paths in S. Estimate the arithmetic mean of $D(S)$ over all non-equivalent acceptable n-point sets S, and also over all finite acceptable sets.

4. What if we replace the condition *ii)* by 'either there is no point $(a_k, b_k) \in S$ such that $a_i < a_k < a_j$ and $b_i < b_k$, or there is no point $(a_k, b_k) \in S$ such that $a_i < a_k$ and $b_i < b_k < b_j$'?

5. Formulate and investigate N-dimensional analogs of the problem, where $N > 2$.

2. **Separating Functions**

Consider n positive integers $a_1, a_2, \ldots, a_n$ which are coprime, *i.e.*, $\gcd(a_1, a_2, \ldots, a_n) = 1$. Let S be the set of all linear combinations of numbers $a_1, a_2, \ldots, a_n$ with *nonnegative* integer coefficients,

$$S = \{x_1a_1 + x_2a_2 + \cdots + x_na_n \;:\; x_i \in \mathbb{Z},\, x_i \geq 0\}\,.$$

1. Show that there exists a number $f \in \mathbb{N}$ such that for any integer $a \geq f$ we have $a \in S$.

Define $F(a_1, a_2, \ldots, a_n)$ as the minimal number $f \in \mathbb{N}$ such that S contains all integers starting from f,

$$F(a_1, a_2, \ldots, a_n) = \min\{f \in \mathbb{N} \;:\; \text{all integers } a \geq f \text{ belong to } S\}\,.$$

We will call F a *separating function*.

2. Case $n = 2$. Find a polynomial $p(x, y)$ with real coefficients such that $F(a_1, a_2) = p(a_1, a_2)$ for every pair of coprime positive integers a_1, a_2.

3. Case $n = 3$.

a) Show that if two of three coprime positive integers a_1, a_2, a_3 are even, then $F(a_1, a_2, a_3)$ is also even.

b) Suppose $a_3 \geq F(\frac{a_1}{d}, \frac{a_2}{d})$ where $d = \gcd(a_1, a_2)$. Prove that

$$F(a_1, a_2, a_3) = d \cdot F(\frac{a_1}{d}, \frac{a_2}{d}) + F(d, a_3).$$

4. Suggest and study a generalization of the question 3.a for $n > 3$.

5. Try to find relations and divisibility rules for $F(a_1, a_2, \ldots, a_n)$.

3. **A Cyclic Inequality**

Let a be a nonnegative real number, and n a positive integer. Denote by

$$A(x) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

the arithmetic mean of an n-tuple $x = (x_1, x_2, \ldots, x_n)$ of positive real numbers. By definition, a *cyclic mean of order k* has the following expression

$$C_{k,a}(x) = \frac{1+a}{n}\left(\frac{x_1^{k+1}}{x_1^k + ax_2^k} + \frac{x_2^{k+1}}{x_2^k + ax_3^k} + \cdots + \frac{x_n^{k+1}}{x_n^k + ax_1^k}\right).$$

1. Show that if $a \geq k-1$ then $C_{k,a}(x) \geq A(x)$ for any n-tuple x of positive real numbers.

2. Prove that if $0 < a < \frac{k-1}{k+1}$ and $n > 1$ then there exist two n-tuples x and y of positive real numbers such that

$$C_{k,a}(x) > A(x) \quad \text{but} \quad C_{k,a}(y) < A(y).$$

3. Investigate the case that $\frac{k-1}{k+1} \leq a < k-1$.

4. Let l be a positive integer. Consider the following function

$$C_{k,l,a}(x) = \frac{1+a}{n}\left(\frac{x_1^{k+l}}{x_1^k + ax_2^k} + \frac{x_2^{k+l}}{x_2^k + ax_3^k} + \cdots + \frac{x_n^{k+l}}{x_n^k + ax_1^k}\right).$$

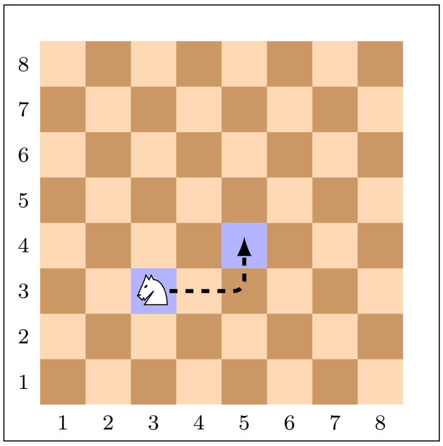
In particular, $C_{k,1,a}(x) = C_{k,a}(x)$. Find the smallest $a > 0$ such that the inequality

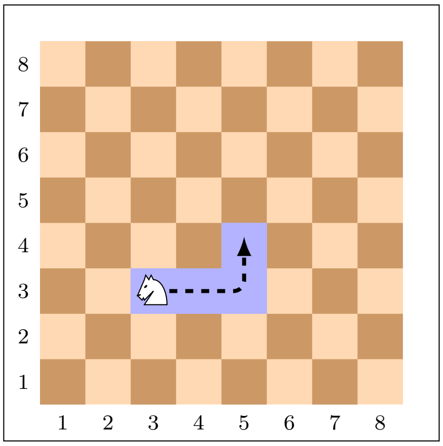
$$C_{k,l,a}(x) \geq (A(x))^l$$

holds for any n-tuple x of positive real numbers.

4. **A Baby Chess**

1. A knight is placed on the square with coordinates (a, b) of an $m \times n$ board. Two students alternately move the knight (following the chess rules) to a square that hasn't been visited yet. The loser is the one who cannot make a move. Find a winning strategy for one of the students.





2. The analogous game with the condition that the knight leaves a 'trace' – four squares that should be considered as visited (see the figure), and it is not allowed to pass through already visited squares.

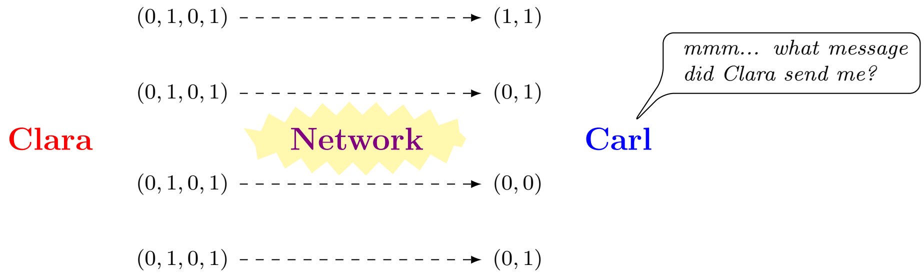
3. Study the same games for other chess pieces: a king, a queen, a rook, a bishop.

5. **A Strange Network**

A *code of length* n is a tuple $A = (a_1, \ldots, a_n)$ of n nonnegative integers. A k-*subcode* of A is a tuple $(a_{i_1}, \ldots, a_{i_k})$, where $1 \leq i_1 < \cdots < i_k \leq n$. For example, $(0, 2, 0, 0)$ is a 4-subcode of $(2, 4, 0, 6, 2, 0, 1, 0)$. Remark that, by definition, the code A has $\frac{n!}{k!(n-k)!}$ *different* k-subcodes even though some of the numbers a_i's might be equal.

Clara sends T copies of a code A of length n to Carl. However, the Network is so strange that instead of receiving T copies of the code A, Carl obtains T different k-subcodes of A. (What Carl knows about the Network is that, as soon as T is not too large, it never gives the same k-subcodes. For instance, if $n = 3$ and Carl receives two 2-subcodes $(1, 2)$ and $(1, 2)$, then he can conclude that A is either $(1, 1, 2)$ or $(1, 2, 2)$.)

At the beginning Carl is given a positive integer n and a number α such that $0 \leq a_i \leq \alpha$ for all $1 \leq i \leq n$. Find or estimate the minimal positive integer $T_{\min} = T_{\min}(n, k, \alpha)$ such that Carl can restore a code A of length n from any $T_{\min}$ different k-subcodes of A. First, investigate particular cases: $k = n - 1$, $k = n - 2$, $k = [n/2]$, $\alpha = 1$, etc.



1. Consider the situation when Carl also knows that all a_i's are distinct.

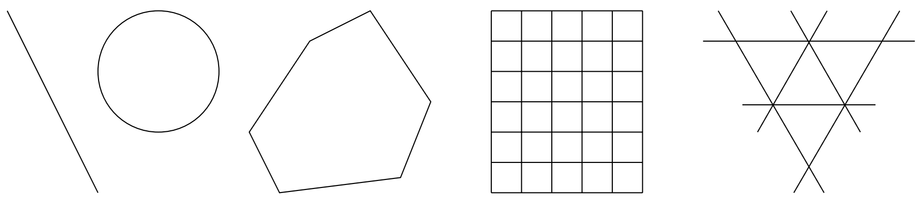
2. Consider the case that the numbers a_i's are not necessary distinct.

3. Suggest and study additional directions of research.

6. Min/Max Questions

1. Let S be an infinite set of points in the plane. Choose $N > 2$ points in S and consider the $\frac{N(N-1)}{2}$ segments connecting them. Denote by $l_{\min}$ and $l_{\max}$ the minimal and the maximal lengths of these segments respectively. What values can the ratio ${}^{l_{\min}}\!/_{l_{\max}}$ take? Investigate the following cases:

a) S is a line, show that $l_{\max} \geq (N - 1) \cdot l_{\min}$,
b) S is a circumference,
c) S is the boundary of a convex polygon,
d) S is an $m \times n$ rectangular grid,
e) S is an infinite triangular grid,
f) S is the whole plane, show for instance that $l_{\max} \geq \sqrt{2} \cdot l_{\min}$ for $N = 4$.



2. Instead of the segments, consider the shortest curves that *lie entirely in the set S*. (For example, if S is a circumference then such a curve connecting two points $x, y \in S$ is the smallest arc of S with endpoints at x and y.) Among the lengths of all shortest curves connecting the chosen N points with each other, let $c_{\min}$ and $c_{\max}$ be the minimal and the maximal ones respectively. Estimate the ratio ${}^{c_{\min}}\!/_{c_{\max}}$. Study the same cases as in the previous question.

3. Consider all triangles with the vertices at three of the N chosen points. Estimate the ratio ${}^{a_{\min}}\!/_{a_{\max}}$ of the minimal area to the maximal area of these triangles.

7. Friendly Polynomials

Let K be some field, and $K[x]$ the set of polynomials with coefficients from K. Given such a polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0 \in K[x]$, the following polynomial

$$P^{(1)}(x) = (P(x))' = na_n x^{n-1} + \cdots + 2a_2 x + a_1$$

is called the (*first*) *derivative* of P. Denote by $P^{(k)}$ the kth successive derivative of P, it is defined recursively by

$$P^{(k)}(x) = \left(P^{(k-1)}(x)\right)' \quad \text{for } k \geq 2.$$

We say that $Q \in K[x]$ *divides* P if there is a polynomial $R \in K[x]$ such that $P = QR$. Two polynomials $P_1, P_2 \in K[x]$ are said to be *coprime* if there is no polynomial $Q \in K[x]$ of degree at least 1 that divides both P_1 and P_2.

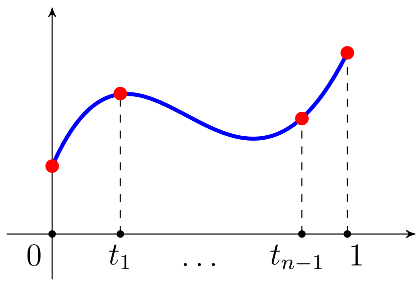
We will call a polynomial $P \in K[x]$ of degree n *friendly* if it is not coprime with any of its derivatives, that is, for all $1 \leq k < n$, the polynomials P and $P^{(k)}$ share a common divisor of degree at least 1.

1. Consider the field $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ of residue classes modulo a prime p. Try to find all friendly polynomials $P \in \mathbb{F}_p[x]$ of degree $n \in \mathbb{N}$.

2. Let $\mathbb{C}$ be the field of complex numbers. Is it true that if a polynomial $P \in \mathbb{C}[x]$ of degree n is friendly then $P(x) = c(x-a)^n$ for some $a, c \in \mathbb{C}$? Study this question for particular values of n (*e.g.,* 2, 3, 4, 5), and also when n is a prime number, a power of a prime number, etc.

8. Points on Curves

1. Let be $f : [0,1] \to \mathbb{R}$ be a continuous function. Find all positive integers n such that there exists a sequence of real numbers $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ for which the expression



$$|f(t_{i+1}) - f(t_i)| + |t_{i+1} - t_i|$$

does not depend on $i = 0, 1, ..., n-1$.
Consider the case that f is

a) a polygonal chain (that is, piecewise linear);
b) a polynomial of degree $k = 2, 3, ...$;
c) a trigonometric function.

2. A sequence $A_0, A_1, \ldots, A_n$ of $n+1$ points in the plane is called α-*tight*, where $\alpha \in \mathbb{R}$, if $A_0 = (0,0)$ and the distance between points A_i and A_j satisfies

$$d(A_i, A_j) \leq \left(\frac{j-i}{n}\right)^{\alpha}, \qquad \text{for all } 0 \leq i < j \leq n.$$

Let $\{A_i = (x_i, y_i)\}_{i=0}^{n}$ be an α-tight sequence of $n+1$ points. A function $F : \mathbb{R}^2 \to \mathbb{R}$ is called C-*fitting* for the sequence $\{A_i\}_{i=0}^{n}$ if the following inequalities hold

$$| F(A_j) - F(A_i) - x_j y_i + x_i y_j | \leq C \cdot \frac{j-i}{n}, \qquad \text{for all } 0 \leq i < j \leq n.$$

Find all (some) numbers C such that for any positive integer n and any α-tight sequence $\{A_i\}_{i=0}^{n}$ there exists a C-fitting function for $\{A_i\}_{i=0}^{n}$. Consider the case that

a) $\alpha = 1$; b) $\alpha = \frac{2010}{2011}$; c) $\alpha = \frac{1}{2}$.

9. A Topological Problem

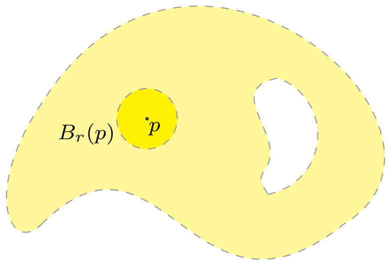
Consider the n-dimensional Euclidean space $\mathbb{R}^n$ with the ordinary metric

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$$

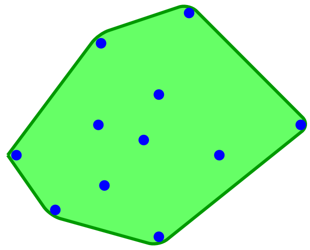
where $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are the coordinates of points $x, y \in \mathbb{R}^n$ respectively. The *open ball* of radius $r > 0$ centered at a point p in $\mathbb{R}^n$, usually denoted by $B_r(p)$, is defined by

$$B_r(p) = \{x \in \mathbb{R}^n \ : \ d(x, p) < r\}.$$

For instance, open balls in the real line $\mathbb{R}$ are open intervals $]a, b[$, open balls in the plane $\mathbb{R}^2$ are circles without their circumferences.



an open set



a convex hull

A set $X \subset \mathbb{R}^n$ is called *open* if, for any point $p \in X$, there exists a real number $r > 0$ such that $B_r(p) \subset X$. A set $Y \subset \mathbb{R}^n$ is called *closed* if its complement $X = \mathbb{R}^n \setminus Y$ is open. A set $Z \subset \mathbb{R}^n$ is called *convex* if, for any two points $p, q \in Z$, the line segment connecting p and q is in Z. Obviously, there exist sets that are neither open nor closed nor convex.

Introduce three operations on the subsets of $\mathbb{R}^n$:

(int) the *interior* of a set $A \subset \mathbb{R}^n$, denoted by $int(A)$, is the union of all open sets X contained in A,

$$int(A) = \bigcup_{\text{open } X \subset A} X,$$

(cl) the *closure* of a set $A \subset \mathbb{R}^n$, denoted by $cl(A)$, is the intersection of all closed sets Y containing A,

$$cl(A) = \bigcap_{\text{closed } Y \supset A} Y,$$

($conv$) the *convex hull* of a set $A \subset \mathbb{R}^n$, denoted by $conv(A)$, is the intersection of all convex sets Z containing A,

$$conv(A) = \bigcap_{\text{convex } Z \supset A} Z.$$

1. Case $n = 1$. What is the maximal number of distinct sets that can be obtained from a set $A \subset \mathbb{R}$ using the operations int and cl?

2. Case $n = 2$.

a) Show that for any set $A \subset \mathbb{R}^2$ such that $int(conv(A)) \neq \emptyset$ we have

$$int(conv(A)) = int(conv(cl(A))).$$

b) What is the maximal number of distinct sets that can be obtained from a set $A \subset \mathbb{R}^2$ using the operations int, cl and $conv$? Consider the cases that $int(conv(A)) = \emptyset$ and $int(conv(A)) \neq \emptyset$.

3. Study the previous questions for $n > 2$.

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