PROBLEMS FOR THE 1^{st} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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Contents

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1. Specular Colourings

1. What is the maximum number of cells of an $m \times n$ grid that can be coloured blue, such that no two blue cells are symmetric with respect to any horizontal or vertical line of the grid?

2. Some cells of an $m \times n$ grid are coloured blue. We call such a colouring *specular* if for any interior horizontal or vertical line of the grid there are two blue cells that are symmetric with respect to this line. Denote by $S(m, n)$ the minimal number of blue cells in a specular colouring of an $m \times n$ grid.

Find $S(m, n)$ or estimate it (give lower and upper bounds).

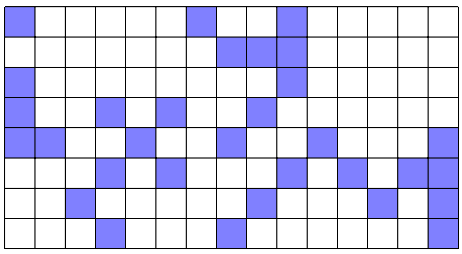


Figure 1. An example of specular, but not minimal, colouring of an 8×15 grid.

3. Formulate and investigate 3-dimensional analogs of the problem.

Date: April 17, 2009.

2. **A Functional Equation**

Let k be a constant real number.

1. Find some (all) functions $f : \mathbb{R} \to \mathbb{R}$ with the property that $f(f(x) + kx) = xf(x)$ for all real numbers x.

2. Find all solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(f(x) + f(y) + kxy) = xf(y) + yf(x), \qquad x, y \in \mathbb{R}. \tag{1}$$

Consider the case when f is (a) a polynomial, (b) a continuous function, (c) an arbitrary function.

3. Let $n > 2$ be a positive integer. Find some (all) functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x_1) + f(x_2) + \ldots + f(x_n) + kx_1x_2...x_n) = x_1f(x_2) + x_2f(x_3) + \ldots + x_nf(x_1)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

4. Suggest and investigate other generalisations of the functional equation (1).

3. **Monotonic Squares**

We say that a positive integer $a = a_ka_{k-1}\ldots a_1a_0$, where $0 \leqslant a_i \leqslant 9$ are the digits of a in base 10, is an *increasing square* if $a = b^2$ for some integer b and $a_k \leqslant a_{k-1} \leqslant \cdots \leqslant a_1 \leqslant a_0$. For instance, $13456 = 116^2$.

If we have the reverse inequalities $a_k \geqslant a_{k-1} \geqslant \cdots \geqslant a_1 \geqslant a_0$, then the square a is called *decreasing*. For instance, $8874441 = 2979^2$.

Let $a = b^2$ and $c = d^2$ be two squares with the following representations in base 10:

$$a = a_ka_{k-1}\ldots a_1a_0,\ b = b_lb_{l-1}\ldots b_1b_0 \quad \text{and} \quad c = c_mc_{m-1}\ldots c_1c_0,\ d = d_nd_{n-1}\ldots d_1d_0.$$

We say that a pair of squares a and c is *ordered*, and write $a \prec c$, if the sequence $a_0, a_1, \ldots, a_{k-1}, a_k$ is a subsequence of $c_0, c_1, \ldots, c_{m-1}, c_m$ and the sequence $b_0, b_1, \ldots, b_{l-1}, b_l$ is a subsequence of $d_0, d_1, \ldots, d_{n-1}, d_n$. For instance, $1156 = 34^2 \prec 111556 = 334^2$.

A set of squares F is called a *family* if any pair of squares from F is ordered.

1. Find infinite families of increasing squares. For instance, $1156 = 34^2 \prec 111556 = 334^2 \prec 11115556 = 3334^2 \prec \ldots$.

2. Is there any infinite family of decreasing squares?

3. How many elements are in a maximal increasing family? For example, can it have exactly (a) one increasing square? (b) two increasing squares? (A *maximal increasing* family F is a family of increasing squares, such that any increasing square a with the property "either $a \prec c$ or $c \prec a$ for all $c \in F$" is already in F.)

3. How many elements can a maximal family of decreasing squares have?

4. Investigate the problem in other bases.

4. **Minimality of Inscribed Polygons**

We say that a polygon P is *inscribed* in a polygon Q if the vertices of P lie on edges of Q, no two on the same edge. (A vertex of an inscribed polygon P is not allowed to coincide with a vertex of the polygon Q.)

1. A triangle T is inscribed in a triangle ABC, so that ABC is divided into four triangles: T_1, T_2, T_3 and T (see the picture).

(a) Is it always true that $area(T) \geqslant \min\{area(T_1), area(T_2), area(T_3)\}$?

(b) Can T have a bisector (median, perimeter, angle, inscribed or circumscribed circle, etc.) smaller than all bisectors (medians, perimeters, angles, inscribed or circumscribed circles, etc.) of the triangles T_1, T_2, T_3?

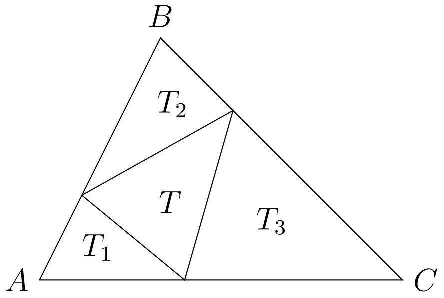


FIGURE 2. A triangle T inscribed in a triangle ABC.

2. A convex polygon $P = P_1P_2 \dots P_m$ is inscribed in a convex polygon $Q = Q_1Q_2 \dots Q_n$, where $3 \leqslant m \leqslant n$, so that Q is divided into $m+1$ parts. Can the polygon P possess a minimality property compared to the parts (for instance, can P have the smallest area, perimeter, angle, diagonal, etc.)?

3. Formulate and investigate 3-dimensional analogs of the problem.

5. **Integer-valued Polynomials**

An *integer-valued polynomial* $q(x)$ is a polynomial taking an integer value $q(n)$ for every positive integer n. Denote by $\mathbb{Q}_0[x]$ the set of all integer-valued polynomials with rational coefficients, that is

$$\mathbb{Q}_0[x] = \{q(x) \in \mathbb{Q}[x] \mid q(n) \in \mathbb{Z}, \forall n \in \mathbb{N}\}.$$

Let p be a prime number and let $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ be the set of residues modulo p.

1. Describe the set of integer-valued polynomials $q(x) \in \mathbb{Q}_0[x]$ with the property $q(n) \equiv 0 \bmod p$ for all $n \in \mathbb{N}$.

2. Let $q(x)$ be a polynomial from $\mathbb{Q}_0[x]$. Define whether the sequence $(q(n) \bmod p)_{n\in\mathbb{N}}$ is periodic and, if it is, find or estimate its period.

3. We say that a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{Z}_p$ is *realisable* if there exists an integer-valued polynomial $q(x) \in \mathbb{Q}_0[x]$ such that $q(n) \equiv \alpha_n \bmod p$ for all $n \in \mathbb{N}$. Describe the set of realisable sequences.

4. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a realisable sequence. Describe the set of integer-valued polynomials $q(x) \in \mathbb{Q}_0[x]$ such that $q(n) \equiv \alpha_n \bmod p$ for all $n \in \mathbb{N}$.

5. Describe the set $\mathbb{Q}_0[x]$.

6. **Pattern Graphs**

Let n be a positive integer. A *pattern of length* n is a two-line table $\begin{matrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{matrix}$, where $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are some rearrangements of the numbers $1, 2, \dots, n$.

Define two operations A and B on patterns as follows

A : replace each number a of the first line with the number that is in the a'th place (from left to right) of the second line,

B : replace each number b of the second line with the number that is in the b'th place (from left to right) of the first line.

We can construct an oriented labelled graph G_n whose vertices are all the patterns of length n, and such that for any two vertices v and w there is an A-arrow (resp. a B-arrow) from v to w if the pattern w is obtained from the pattern v by applying the operation A (resp. the operation B).

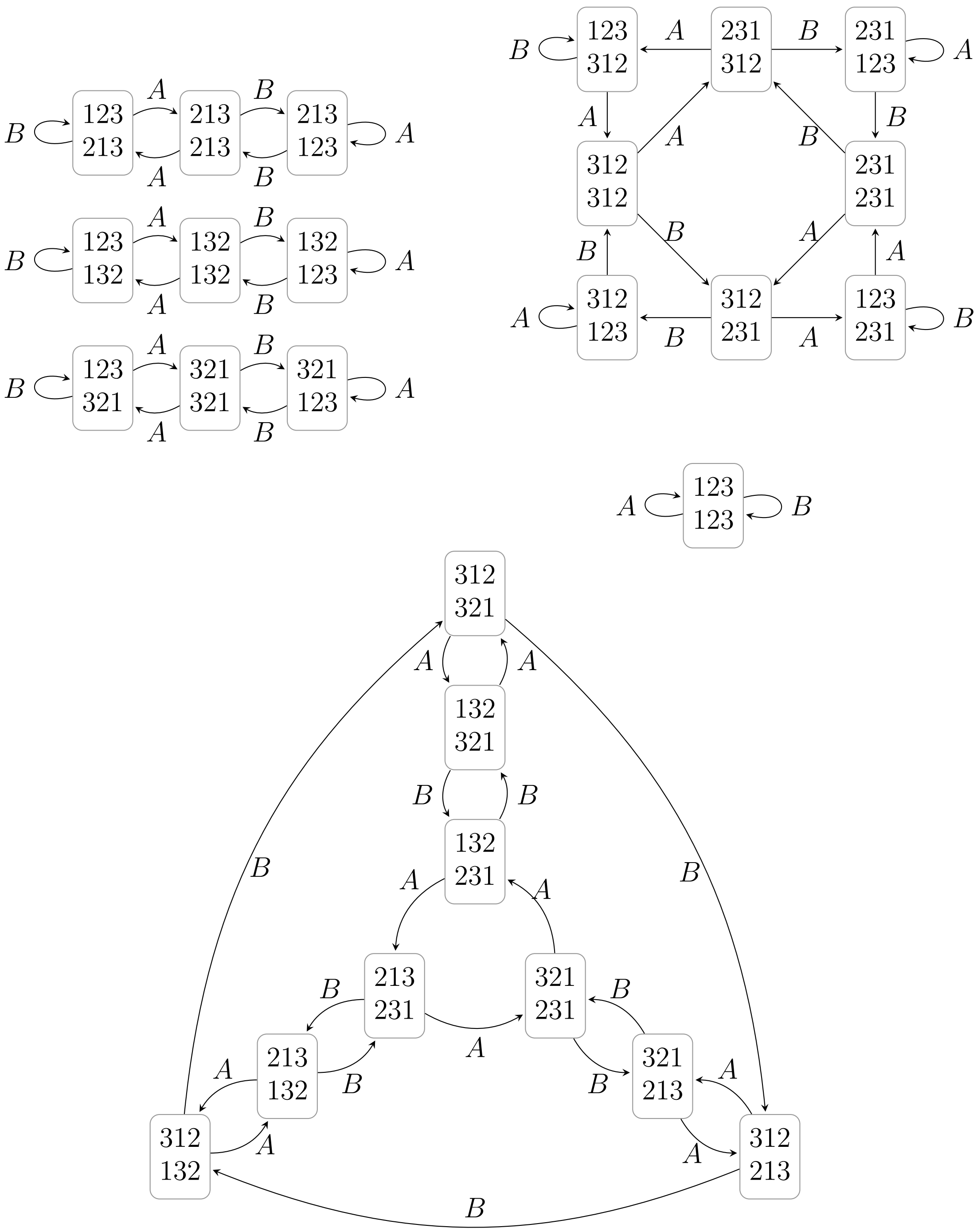


FIGURE 3. The graph G_3 has 7 connected components (the 7th one with 9 vertices is not shown on the picture).

1. Can the pattern $\begin{array}{cccccc} 2 & 1 & 3 & 4 & \dots & n-1 & n \\ 2 & 3 & 4 & 5 & \dots & n & 1 \end{array}$ be obtained from the pattern $\begin{array}{ccccccc} 2 & 3 & 1 & 4 & \dots & n-1 & n \\ 2 & 3 & 4 & 5 & \dots & n & 1 \end{array}$ using the operations A and B?

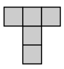
2. How many connected components of the graph G_n have exactly (a) 2 patterns? (b) 3 patterns?

3. Denote by g_n the number of connected components of the graph G_n. Find g_n (give a formula) or estimate it (give lower and upper bounds).

4. Study geometric properties of the connected components of G_n. Can they be drawn nicely in the plane: without intersections of the edges, symmetrically, etc.?

5. As a generalisation one could consider three-line patterns with $3 \cdot 2 = 6$ operations on them. Investigate this generalisation.

7. **Placements of Pentominoes**

1. Given an $m \times n$ rectangle, denote by $T(m, n)$ the minimum number of non-overlapping pentominoes  that must be placed (along the grid lines) so that there is no place on the free cells for another pentomino? Find or estimate the number $T(m, n)$ and give an algorithm for constructing suitable placements.

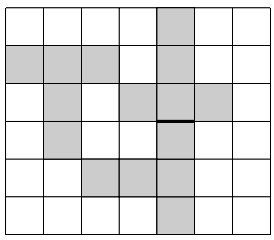
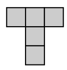


FIGURE 4. A placement of pentominoes on a 6×7 rectangle: $T(6, 7) = 3$.

2. Two players alternately place pentominoes  on the free cells of an $m \times n$ rectangle, along the grid lines. The loser is the one who cannot place a pentomino. Does any player have a winning strategy?

3. Study the previous questions for other polyominoes.

8. **Positivity of Symmetric Polynomials**

A polynomial $P(x, y)$ with real coefficients is *symmetric* if the equality $P(x, y) = P(y, x)$ holds for all $x, y \in \mathbb{R}$.

1. Let $P(x, y) = x^3 + ax^2y + axy^2 + y^3$ be a symmetric polynomial of degree 3. Prove that $P(x, y) > 0$ for all $x, y > 0$ if and only if $a > -1$.

2. Let $P(x, y) = x^4 + ax^3y + bx^2y^2 + axy^3 + y^4$ be a symmetric polynomial of degree 4.

(a) Prove that $P(x, y) > 0$ for all $x, y > 0$ if and only if $a < -4$, $b > \frac{a^2+8}{4}$ or $a \geqslant -4$, $b > -2a - 2$.

(b) Prove that $P(x, y) > 0$ for all $x, y \neq 0$ if and only if $|a| > 4$, $b > \frac{a^2+8}{4}$ or $|a| \leqslant 4$, $b > 2|a| - 2$.

3. Let $P(x,y)$ be one of the following symmetric polynomials:

$$x^5 + ax^4y + bx^3y^2 + bx^2y^3 + axy^4 + y^5,$$
$$x^6 + ax^5y + bx^4y^2 + cx^3y^3 + bx^2y^4 + axy^5 + y^6,$$
$$x^7 + ax^6y + bx^5y^2 + cx^4y^3 + cx^3y^4 + bx^2y^5 + axy^6 + y^7.$$

Find necessary and sufficient conditions on the coefficients such that $P(x,y) > 0$ for all (a) $x, y > 0$, (b) $x, y \neq 0$.

4. Find sufficient conditions for a homogeneous symmetric polynomial $P(x,y)$ of degree $n > 7$ to take positive values for all (a) $x, y > 0$, (b) $x, y \neq 0$.

5. Using methods developed in the previous questions, give necessary and sufficient conditions for a non-homogeneous symmetric polynomial of two real variables to be positive.

9. **Good Numbers**

Any rational number x may be expressed as a continued fraction

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_m}}}}$$

where a_0 is the integer part of x, and the numbers $a_1, a_2, \ldots, a_m$ are positive integers called *partial quotients* of x. We will also write $x = [a_0; a_1, a_2, \ldots, a_m]$.

1. Find all numbers $n > 2$ that can be expressed as the sum of two positive integers $n = a+b$ so that $a < b$ and the continued fraction for a/b has all its partial quotients equal to 1. For example, $13 = 5 + 8$ and $5/8 = [0; 1, 1, 1, 1, 1]$.

2. A number $n > 2$ is called *2-good* if for some positive integers $a < b$ we have $n = a + b$ and the partial quotients of a/b are equal to 1 or 2. For example, $11 = 4+7$ and $4/7 = [0; 1, 1, 2, 1]$.

(a) Are there infinitely many odd numbers that are *not* 2-good?
(b) Is it true that any even positive integer, greater than 6, is the sum of two distinct odd 2-good numbers? If it is not, find all even numbers with this property.
(c) Describe the set of all 2-good numbers.

3. In general, a number n is called *k-good* if it can be expressed as the sum of two positive integers $n = a+b$ so that $a < b$ and the continued fraction for a/b has all its partial quotients not greater than k.

Does there exist a positive integer K such that all positive integers $n > 2$ are K-good?

4. Describe the set of numbers n with the property: there exist two coprime positive integers $b > a$ such that $n = b + a$ or $n = b - a$ and the partial quotients of a/b are equal to 1 or 2.

5. Suggest and study additional directions of research

10. **Centro-Symmetric Shadows**

A set of points in the plane is said to be *centro-symmetric* if it has a centre of symmetry. For example, the vertices of a square form a centro-symmetric set. (Recall that a *centre of* *symmetry* of a set S in the plane is a point c with the property that for any point $p \in S$ there exists a point $p' \in S$ such that p and p' are equidistant from c and lie on a line passing thought c.)

Given a set of points, its *shadow* on a line is its orthogonal projection onto this line.

1. Let $n \geqslant 3$ be a positive integer. Denote by $k(n)$ the minimum positive integer k with the following property:

> for any set S of n points in the plane, if there exist k lines, no two parallel, such that for each line the shadow of S on this line is centro-symmetric, then the initial set S is also centro-symmetric.

Find or estimate the number $k(n)$.

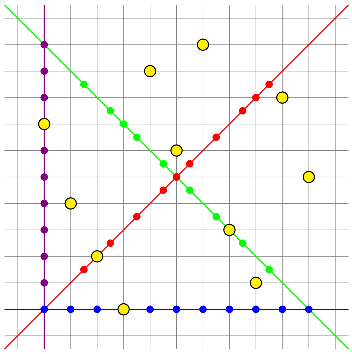


FIGURE 5. A set of 11 points with 4 centro-symmetric shadows: $k(11) > 4$.

2. Formulate and study 3-dimensional analogs of the problem.

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