PROBLEMS FOR THE 10^{TH} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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Contents

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Notation

$\mathbb{N} = \{1, 2, 3, \ldots\}$	set of natural numbers (positive integers)
$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$	ring of residue classes modulo n
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	sets of integer, rational, real and complex numbers
$\mathbb{R}_+$	set of positive real numbers
$\mathbb{F}_q$	finite field of size $q = p^k$
$M \otimes N$	mean-product of two functions M and N
$\gcd(x_1, \ldots, x_n)$	greatest common divisor of $x_1, \ldots, x_n$
$\mathbb{P}[.]$ and $\mathbb{E}[.]$	probability and expectation
$\det(A)$ or $\|A\|$	determinant of a matrix A
$\|x\|$	absolute value of x
$\#M$ or $\|M\|$	cardinality of a set M
$\sup\limits_{x \in M} f(x)$	supremum of a set $\{f(x) \mid x \in M\}$

1. Maximal Cardinalities of Bounded-angled Sets

1. Let S be a set of vectors in $\mathbb{R}^3$.

a) Suppose that the angle between every two vectors in S is strictly greater than $\frac{\pi}{2}$. Find the maximal possible value of $|S|$.

b) Suppose that the angle between every two vectors in S is at least α, where $\alpha \in (0, \pi)$. Show that $|S| \leq \frac{2}{1-\cos\frac{\alpha}{2}}$. When does equality hold?

Date: March 21, 2018.

Let d be a positive integer and $\alpha \in (0, \pi)$ – a real number. Let $f_\alpha(d)$ be the maximal cardinality of a set of points in $\mathbb{R}^d$ such that any angle formed by three of the points is *strictly less than* α. Similarly, let $g_\alpha(d)$ be the maximal cardinality of a set of points in $\mathbb{R}^d$ such that any angle formed by any three of the points is *less than or equal to* α.

2. Prove that for any fixed d and α the values $f_\alpha(d)$ and $g_\alpha(d)$ are indeed finite.

3. Find $g_\alpha(d)$ and describe all sets for which the value is attained when:

a) $\alpha < \frac{\pi}{3}$
b) $\alpha = \frac{\pi}{3}$
c) $\alpha = \frac{\pi}{2}$
d) $\alpha = \frac{2\pi}{3}$
e) $\alpha = \frac{3\pi}{4}$

4.

a) Prove the inequality $f_{\frac{\pi}{2}}(d+2) \geq 2f_{\frac{\pi}{2}}(d)$.
b) Prove the inequality $f_{\frac{\pi}{2}}(d) \geq 2^{d-1} + 1$.
c) Try to give better lower and upper bounds for $f_{\frac{\pi}{2}}(4)$.

5. Give (as best as you can) a lower bound on $f_\alpha(d)$ for (at least some) values $\alpha > \frac{\pi}{2}$. Try also to establish inequalities of the form $f_\alpha(d+k) \geq c f_\alpha(d)$ for suitable $k > 0$ and $c > 1$, possibly depending on $\alpha > \frac{\pi}{2}$.

6.

a) Find (as best as you can) $\varepsilon > 0$ such that for all $\alpha \in (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2})$ the value $f_\alpha(d)$ grows exponentially with respect to d or prove that no such ε exists. In case it exists, give an explicit lower bound for $f_\alpha(d)$ which shows that.
b) Find (as best as you can) $\varepsilon > 0$ such that for all $\alpha \in (\frac{\pi}{3}, \frac{\pi}{3} + \varepsilon)$ the value $f_\alpha(d)$ grows polynomially, of degree at most $m \geq 1$, with respect to d or prove that no such ε exists. In case it exists, give an explicit upper bound for $f_\alpha(d)$ which shows that. You may start with the case $m = 1$.
c) Find the angle α_0 such that $f_\alpha(d)$ is an exponential with respect to d for $\alpha > \alpha_0$ and a polynomial for $\alpha < \alpha_0$. What happens at $\alpha = \alpha_0$?
d) Investigate b) and c) for $g_\alpha(d)$.

7. Fix d and let α vary. For what values $\beta \in [\frac{\pi}{3}, \pi)$ (if any) are the functions f_α and g_α continuous at β? Differentiable at β?

8. Suggest and study additional directions of research.

2. **Coloring Graphs**

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. In a *red-blue edge coloring* of the graph G, every edge of G is colored red or blue. Similarly, in a *red-blue vertex coloring* of the graph G, every vertex of G is colored red or blue.

Let r and b be positive integers. We write $G \to (r, b)^e$ ($G \to (r, b)^v$) if for every red-blue edge coloring (red-blue vertex coloring) of G, there is either a complete subgraph isomorphic to K_r, all of whose edges (vertices) are colored red, or a complete subgraph isomorphic to K_b, all of whose edges (vertices) are colored blue. For example, it is well-known that $K_6 \to (3, 3)^e$, or, in other words, that any red-blue edge coloring of K_6 implies that one can find either a red or a blue K_3 (= a triangle) in K_6.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \subseteq G$. Note that a graph G is a subgraph of itself. Let p be a positive integer. Define

$X_e(r,b,p) = \{G : G \to (r,b)^e \text{ and } K_p \nsubseteq G\}$ and $X_v(r,b,p) = \{G : G \to (r,b)^v \text{ and } K_p \nsubseteq G\}$. In other words, $X_e(r,b,p)$ (resp., $X_v(r,b,p)$) is the set of all graphs G, such that:

- for every red-blue edge coloring (resp., red-blue vertex coloring) of G one can either find a red K_r, or a blue K_b in G;
- and G does not contain a complete graph K_p as a subgraph.

For example, we see that $K_6 \in X_e(3,3,p)$ for all $p > 6$, but $K_6 \notin X_e(3,3,6)$ exactly because of the second condition in the above definition.

1. Prove that the set $X_e(3,3,6)$ is not empty.

2.

a) Is it true that $X_e(r,b,p)$ and $X_v(r,b,p)$ are non-empty sets if $p > \max\{r,b\}$?
b) What happens if $p \leq \max\{r,b\}$?
c) Let $F_e(r,b,p) = \min\{|V(G)| : G \in X_e(r,b,p)\}$ and $F_v(r,b,p) = \min\{|V(G)| : G \in X_v(r,b,p)\}$. If $p > r+b-1$, prove that $F_v(r,b,p) = r+b-1$.

3.

a) Find (or give bounds on) the numbers $F_e(3,3,6)$, $F_e(3,5,14)$ and $F_e(4,4,18)$.
b) Prove that $F_e(3,3,5) \leq F_v(3,3,4)+1$.
c) Find (or give bounds on) $F_v(3,3,4)$ and $F_e(3,3,5)$.

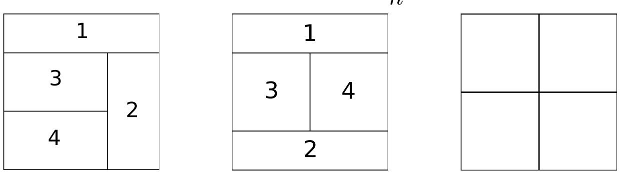
4. Let $n \in \mathbb{N}$. Denote $X'_e(r,b,p,n) = \{G : G \in X_e(r,b,p) \text{ and } |V(G)| = n\}$.
a) Determine $|X'_e(3,3,5,F_e(3,3,5))|$.
b) Let $G \in X'_e(3,3,5,F_e(3,3,5))$. Prove that $\chi(G) = 6$, where $\chi(G)$ is the chromatic number of G.
c) Determine or find bounds of $\min\{|E(G)| : G \in X'_e(3,3,5,F_e(3,3,5))\}$ and $\max\{|E(G)| : G \in X'_e(3,3,5,F_e(3,3,5))\}$.

5. Study $F_e(r,b,p)$, $F_v(r,b,p)$ and $X'_e(r,b,p,n)$ for other values of r, b, p, n.

3. **Cutting a Rectangle**

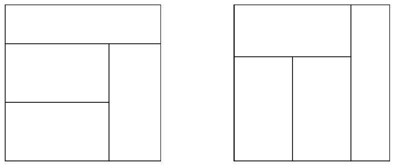
Let's consider a rectangle R. We define a cut of the rectangle R into n pieces to be a set of n rectangles each with area $\frac{1}{n}$ of the area of R, whose union is R itself. Let $C_{R,n}$ be the set of all cuts of R into n pieces.

1. Let $S_{R,n} = \{x \in C_{R,n} |\ \exists$ ordering of the elements of x : $r_1, r_2, \ldots r_n$ such that $\forall i, \bigcup_{j=i}^{n} r_j$ is a rectangle$\}$. Informally, $S_{R,n}$ is the set of cuts which can be achieved by always cutting a whole side from the biggest remaining part of R with area $\frac{1}{n}$ the area of R. For example:



All cuts above are from $C_{R,4}$, but only the two on the left are from $S_{R,4}$ with ordering of elements as shown on the picture (this also corresponds to the order for cutting the pieces for the informal definition). For every ordering of the cut on the right $\bigcup_{j=2}^{4} r_j$ is not a rectangle, thus it's not in $S_{R,4}$.
Provide a formula for $|S_{R,n}|$ with respect to n.

2. Let R be a square. Find the number of different cuts from $S_{R,n}$ with respect to rotations and symmetries. For example, these two cuts should be counted as one:



3. Consider $|C_{R,n}|$ with respect to n. Provide lower and upper bounds of $|C_{R,n}|$ (as strong as you can).

4. Evaluate the number of different cuts from $C_{R,n}$ with respect to rotations and symmetries, where R is a square.

5. Consider 3 and 4's analogical questions for cuts of parallelepipeds in 3 dimensions and generalize for k dimensions.

4. **Edge Realizations of Graphs**

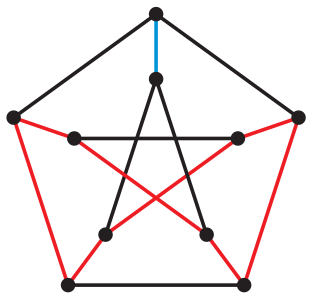
Let G be a simple graph with the set of vertices $V(G)$ and the set of edges $E(G)$. Recall that *the order of* G is equal to the number of its vertices. A graph H is called a *subgraph of* G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Any set of edges $E' \subseteq E(G)$ *implies a subgraph* H, whose vertices are all end points of E' and $E(H) = E'$.

The *distance* $d(u, v)$ between vertices u and v of a connected graph G is the number of edges in the shortest path connecting u and v (we assume that $d(u, u) = 0$). Let uv be some edge of G. The set of edges

$$\{xy \in E(G) \mid \min\{d(x, u), d(x, v), d(y, u), d(y, v)\} = 1\}$$

is called the *edge neighborhood* of the edge uv.

Fix a certain graph H. A graph G is called an *edge realization* of H, if the edge neighborhood of each its edge induces a subgraph isomorphic to the graph H. In this case we call the initial graph H to be *edge realizable*. For example, one can check that the Petersen graph is an edge realization of the simple cycle C_8 of order 8 (see the picture where the red subgraph is a cycle C_8 induced by the edge neighborhood of the blue edge).



1. Find all graphs of order not greater than 4 which are edge realizable and find their edge realizations.

2. Let G_1 and G_2 be two connected edge realizations of some graph H, and let G_2 be a subgraph of G_1. Does this necessarily imply that G_1 and G_2 coincide? If not, which conditions would suffice to ensure that?

3. Let G be a connected edge realization of the graph H. Prove that:

a) if H is not a complete graph, then $\Delta(H) \leq \Delta(G) \leq \Delta(H) + 1$;
b) if $\delta(H) \geq 2$, then $\delta(G) \geq \delta(H) + 1$.

Here $\Delta(G)$ and $\delta(G)$ denote the maximal and the minimal degree of vertices of G respectively.

4. Prove that for any edge realizable graph H of order n we have $\alpha(H) \geq n - 2\Delta(H)$, where $\alpha(H)$ is the vertex independence number of the graph H. (*The vertex independence number of a graph* is the maximal number of vertices in the graph such that no two such vertices are connected by an edge.)

5. Let $\Delta_r = \{H \text{ is a graph} \mid \Delta(H) \leq r\}$ and let $\mathcal{R} = \{H \text{ is a graph} \mid H \text{ is edge realizable}\}$. Prove that for a fixed $r \geq 1$ the set $\mathcal{R} \cap \Delta_r$ is finite.

6. Classify edge realizable graphs within the following families of graphs: (a) graphs with dominating vertices; (b) complete bipartite graphs; (c) complete k-partite graphs with $k > 2$; (d) simple cycles; (e) simple chains; (f) graphs of maximal degree 2.
Provide also the complete description of their connected edge realizations.

7. Suggest your own generalizations of this problem and investigate them.

5. **Expansion in Graphs**

Let b, d, α be positive constants. We consider a connected undirected graph $G = (V, E)$, where V and E are the sets of vertices and edges respectively. For any nonempty proper subset $A \subset V$ denote by ∂A the set of edges $(v_i, v_j) \in E$ with $v_i \in A$, $v_j \in \overline{A} = V \setminus A$. We say that the graph G admits (b, α)-*weak expansion* if $|\partial A| \geq b|A|^\alpha$ for any nonempty proper subset $A \subset V$ with $|A| \leq |V|/2$. If the graph G admits (b, α)-weak expansion and any vertex from V has degree at most d, we say that the graph G admits (b, α, d)-*expansion*. Graph G is called d-regular if degree of any vertex of G is equal to d. Suppose there is given a sequence of connected undirected graphs $G_n = (V_n, E_n)$ such that $|V_n| \to \infty$ as $n \to \infty$. We say that the sequence G_n is (b, α)-*weak expander* if any graph G_n admits (b, α)-weak expansion. Analogously we say that the sequence G_n is (b, α, d)-*expander* if any graph G_n admits (b, α, d)-expansion.

Below are the main tools from theory of expanders. To learn more about the topic, one can also try *Expander Families and Cayley Graphs: A Beginner's Guide* by M.Krebs and A.Shaheen.

The Cheeger constant is defined as

$$h(G) = \min_{A \subset V, 0 < |A| \leq \frac{|V|}{2}} \frac{|\partial A|}{|A|}.$$

The spectral gap σ of a graph G is the smallest positive eigenvalue of its Laplacian matrix.

Let G be a connected undirected d-regular graph, then the Cheeger inequality states

$$\frac{\sigma}{2} \leq h(G) \leq \sqrt{2\sigma d}.$$

Let G be a connected undirected d-regular graph on n vertices. Corollary from the expander mixing lemma gives the inequality

$$\left| |\partial A| - \frac{d|A||\overline{A}|}{n} \right| \leq d\sqrt{|A||\overline{A}|}.$$

1. For any fixed positive integer d construct a graph $G = (V, E)$, which admits (b, α, d)-expansion, where:

a) $\alpha = 2$, $b = 1/2$;
b) $\alpha = 1$, $b = 1$, $|V| \geq 2^d$.

2. Find the greatest positive α (or show that such α does not exist) such that one can find positive constant b such that family of graphs G_n, $n \in \mathbb{N}$, is (b, α, d)-expander, where:

a) G_n is the path on n vertices ($d = 2$);
b) G_n is the graph of the $(n \times n)$-rectangular grid on the Euclidean plane ($d = 4$);
c) G_n is the graph of the $(n \times n \times n)$-grid in the three-dimensional Euclidean space ($d = 6$).

3. Find the set of positive α such that one can find positive b, d and a family of d-regular graphs G_n, $n \in \mathbb{N}$, which is (b, α, d)-expander. Investigate the same problem for the graphs, which are not d-regular in general case.

4. Let's consider the set S_n of all permutations of the set $\{1, \ldots, n\}$. Define graph T_n as follows: the vertices of T_n are the elements of S_n, any two vertices are connected by an edge if and only if for the corresponding permutations s, t the element $s^{-1} \circ t$ is a transposition. Prove that the sequence T_n is $(1/2, 1)$-weak expander.

5. Construct the graph B_n by using the same rules as in the previous question, but getting only transpositions $(i-1, i)$, $i = 1, \ldots, n$, instead of all transpositions. Show that the sequence B_n is not $(b, 1)$-weak expander for any positive b. Find the largest $\alpha > 0$ such that for any $\alpha_1 < \alpha$ one can find $b_1 > 0$ such that the sequence B_n is (b_1, α_1)-weak expander.

6. Suggest and investigate your own directions of this problem.

6. Generalized Commuting Graphs of Finite Groups

Throughout this problem S_n and A_n are the symmetric and the alternating groups on n letters respectively, $D_n = \langle x, y \mid x^n = y^2 = (xy)^2 = 1 \rangle$ is the dihedral group of order $2n$; $|G|$ is the order of a group G; $G_1 \cong G_2$ means that groups G_1 and G_2 are isomorphic and $\Gamma_1 \simeq \Gamma_2$ means that graphs Γ_1 and Γ_2 are isomorphic.

$$\mathrm{Z}(G) = \{x \in G \mid xy = yx \, \forall y \in G\}$$

is the center of a group G. The commuting graph $\Gamma(G)$ of G is the graph, whose set of vertices is $G \setminus \mathrm{Z}(G)$ and two vertices a and b are connected if $ab = ba$. It is clear that (a, b) is an edge of $\Gamma(G)$ iff (b, a) is also an edge. So we can consider this graph as undirected. For example the commuting graph of the symmetric group of degree 3 contains five vertices: (123), (132), (12), (13) and (23); and one edge: ((123), (132)).

1.

a) Are $\Gamma(S_4)$, $\Gamma(A_4)$, $\Gamma(D_4)$ connected?
b) Compute diameters of connected components of $\Gamma(S_4)$, $\Gamma(A_4)$, $\Gamma(D_4)$.
c) Let $K \in \{S_4, A_4, D_4\}$ and G be a finite group with $\Gamma(G) \cong \Gamma(K)$. Does $|G| = |K|$? Does $G \cong K$?

2. Let

$$L = \{D_n \mid n \in \mathbb{N}\} \cup \{A_n \mid n \in \mathbb{N}\} \cup \{S_n \mid n \in \mathbb{N}\}$$

a) For which groups $G \in L$ the commuting graph $\Gamma(G)$ is connected?
b) Compute or give bounds on diameters of connected components of $\Gamma(G)$, $G \in L$.
c) Find all $K \in L$ such that if G is a finite group with $\Gamma(G) \simeq \Gamma(K)$, then $|G| = |K|$.
d) Find all $n \in \mathbb{N}$ such that if G is a finite group with $\Gamma(G) \simeq \Gamma(D_n)$, then $G \cong D_n$.

3. Let $k, m \in \mathbb{N}$ and

$$\mathrm{Z}(G, k, m) = \{x \in G \mid x^k y^m = y^m x^k \, \forall y \in G\}.$$

Let $\Gamma(G, k, m)$ be a (directed) graph of G, whose set of vertices is $G \setminus \mathrm{Z}(G, k, m)$ and there is an edge from a to b if $a^k b^m = b^m a^k$.

a) Is it true that (a, b) is an edge of $\Gamma(G, k, m)$ in the general if and only if (b, a) is also an edge?
b) For which $(k, m) \in \mathbb{N}^2$ we have $\Gamma(S_5, k, m) = \Gamma(S_5, 1, 1)$, i.e.

$$V(\Gamma(S_5, k, m)) = V(\Gamma(S_5, 1, 1)) \text{ and } E(\Gamma(S_5, k, m)) = E(\Gamma(S_5, 1, 1))?$$

c) Does the answer of item 3b change if we replace $\Gamma(S_5, k, m) = \Gamma(S_5, 1, 1)$ by $\Gamma(S_5, k, m) \simeq \Gamma(S_5, 1, 1)$?
d) For which $(k_1, m_1, k_2, m_2) \in \mathbb{N}^4$ we have $\Gamma(S_5, k_1, m_1) \subseteq \Gamma(S_5, k_2, m_2)$, i.e.

$$V(\Gamma(S_5, k_1, m_1)) \subseteq V(\Gamma(S_5, k_2, m_2)) \text{ and } E(\Gamma(S_5, k_1, m_1)) \subseteq E(\Gamma(S_5, k_2, m_2))?$$

e) How many different elements are contained in the set $\{\Gamma(S_5, k, m) \mid (k, m) \in \mathbb{N}^2\}$?
f) Study the same questions for other groups in L.

4. For a given $(k, m) \in \mathbb{N}^2$ study the analogues of questions from 1 and 2 for $\Gamma(G, k, m)$.

7. Inhomogeneous Numbers

Let A be a subset of a commutative ring. A number x is said *inhomogeneous in A*, when there exist some $a, b \in A$ such that $x = a + b = ab$. We denote by I_A the set of inhomogeneous numbers in A.

1. For a given number x, find some conditions that are equivalent to $x \in I_A$, possibly with some conditions on A.

2. What are $I_{\mathbb{Z}}$, $I_{\mathbb{C}}$, $I_{\mathbb{R}}$, $I_{\mathbb{Q}}$? If you notice a common pattern, generalize it to other families of commutative rings.

3. What is $I_{\mathbb{Z}_n}$ for

a) $n = p$, p prime;
b) $n = p^2$, p prime;
c) $n = p^d$, p prime, $d > 2$;
d) any $n \in \mathbb{N}$?

4. A subset S of A is said *a suffi-A* if for every a, b in A, we have $a + b \in S$ or $ab \in S$. In particular, if there are some $a, b \in A$ such that both $a + b$ and ab are not in A, then there exist no suffi-A. How is the notion of a suffi-A related to the above notion of an inhomogeneous number?

5. What is the smallest size of a suffi-A in a some suitable measure μ on A for

a) $A = \mathbb{R}$, μ is the Lebesgue measure (in this case, maximize the size of the complement);
b) $A = \mathbb{N}$, μ is the natural (or asymptotic) density;
c) $A = \mathbb{N}\backslash\{0, ..., k\}$, μ is as in (b);
d) $A = Z_p$, p prime, μ is the cardinality of a subset?

6. A subset S of A is said *an opti-A*, if it is a suffi-A minimal for inclusion. Investigate properties of a sequence S_n such that for any n, S_n is an opti-S_{n-1}. First, find out if any $(S_0, \dots, S_n)$ be continued. In particular, find sufficient conditions on S_0 for the global intersection $\cap_{i=1}^{n} S_i$ to be necessarily an *opti-itself*, and different types of counterexamples. Consider the case of $S_0 = \mathbb{N}, \mathbb{Z}, \mathbb{R}, \dots$.

7. Investigate other related questions such as, for example,

a) find asymptotic bounds on the size of S_n;
b) generalize to S_0 being a set with two commutative (and maybe associative) operations.

8. Mystic Powers of Two

Consider *an initial data* $\{(a_i, b_i)\}_{i=1}^{k}$ of integer pairs satisfying $0 \leq a_i \leq b_i$ and the corresponding sets $P_1, \dots, P_k$ consisting of powers of two

$$P_i := \{2^{a_i}, 2^{a_i+1}, \dots, 2^{b_i}\}.$$

A collection of strictly increasing functions $\varphi_i\colon P_i \longrightarrow \{1, 2, \ldots, m\}$ is called *a monotonic m–coloring* of the initial data. In other words, we color numbers $2^{a_i}, 2^{a_i+1}, \ldots, 2^{b_i} \in P_i$ by "colors"

$$1 \le \varphi_i(2^{a_i}) < \varphi_i(2^{a_i+1}) < \cdots < \varphi_i(2^{b_i}) \le m.$$

The type of a monotonic m–coloring is a sequence $(N_1, N_2, \ldots, N_m)$, where N_i is the sum of all powers of two colored with color i. In this problem the most interesting types are also powers of two. For example, take the initial data $\{(0,1),(1,2)\}$. It defines the subsets $P_1 = \{1,2\}$ and $P_2 = \{2,4\}$. If we color the elements of P_1 with colors 1 and 2, and the elements of P_2 with colors 2 and 3, we get the 3-coloring of type $(1,4,4)$.

A sequence $s = (2^{n_1}, 2^{n_2}, \ldots, 2^{n_m})$ is said to be *mystic*, if for any $k \ge 1$ and for any initial data $\{(a_i,b_i)\}_{i=1}^k$ there exists only even number of monotonic m–colorings of type s.

1. Prove that there exists only one mystic sequence (of powers of two) of length two.

2. A mystic sequence of powers of two is called *minimal* if $(2^{n_1}, 2^{n_2}, \ldots, 2^{n_{m-1}})$ and $(2^{n_2}, \ldots, 2^{n_m})$ are not mystic. Prove that the following sequences are mystic: $(1,1)$, $(1,2,2)$, $(2,2,1)$, $(2,2,2,2)$, $(1,2,1,2,1)$, $(2,1,2,1,2)$. Which of them are minimal? Prove that there exists no minimal mystic sequences with components $1's$ or $2's$ other than those above.

3. Find all mystic sequences of lengths $3, 4, 5$.

4. Prove that sequences $(4,4,4,4,4,4)$ and $(8,8,8,8,8,8,8,8)$ are mystic and minimal.

5. Let $s = (2^{n_1}, \ldots, 2^{n_m})$.

a) Prove that if s is mystic then for any $n \ge 0$ sequences $(2^n, 2^{n_1}, \ldots, 2^{n_k})$ and $(2^{n_1}, \ldots, 2^{n_k}, 2^n)$ are mystic too.

b) Prove that s is mystic if and only if for any $k \ge 1$ and for any initial data $\{(a_i,b_i)\}_{i=1}^k$ satisfying $(a_i,b_i) \ne (a_j,b_j)$ for $i \ne j$ there exists only even number of monotonic m–colourings of type s.

c) Prove that if s is mystic then $n_i = n_j$ for some $i < j$.

d) Assume s is mystic. Prove that if a maximal power of two in s occurs only once then one can delete it and get a mystic sequence of length $m-1$.

e) Prove that there exists $r \ge 1$ such that $s^r := (2^{n_1}, \ldots, 2^{n_m}, \ldots, 2^{n_1}, \ldots, 2^{n_m})$ is mystic. Try to find best possible upper bound for r.

6. Deduce for which n the sequence $(2^n, 2^n, \ldots, 2^n)$ of length $2n+1$ is mystic.

7. Prove that the sequence $(2^n, 2^n, \ldots, 2^n)$ of length $2n+2$ is mystic and minimal.

8. Describe other classes of mystic sequences, not necessarily consisting only of powers of two.

9. On Some Sequences Generated by a Function

Let f be a real valued function defined on the interval $(0;\infty)$. Denote by $L(f)$ be the set of all sequences of real numbers (x_n), $n \in \mathbb{N}$, defined as $x_n = f(nx)$ for some $x \in (0;\infty)$.

1.

a) Is it true that for any sequence (x_n) of real numbers there exists a function f continuous on the interval $(0;\infty)$, such that $(x_n) \in L(f)$?

b) A function f is called *nonexpanding*, if $|f(x) - f(y)| \le |x - y|$ for any $x, y \in (0;\infty)$. Answer item (a) for nonexpanding functions f.

c) Does there exist a function f continuous on the interval $(0;\infty)$, such that the set $L(f)$ coincides with the set of all possible sequences of real numbers?

2. Let as above f be a real function continuous on the interval $(0;\infty)$. Are the following statements equivalent:

(i) the function f is bounded from above on $(0;\infty)$;
(ii) each sequence $(x_n) \in L(f)$ is bounded from above?

3.

a) Is it true that the function f is increasing (strictly increasing) if and only if each sequence $(x_n) \in L(f)$ is increasing (strictly increasing)?
b) Is it true that the function f has finite variation on $(0;\infty)$ if and only each sequence $(x_n) \in L(f)$ has finite variation?
c) Investigate if each sequence $(x_n) \in L(f)$ can be periodic. How this is connected to the periodicity of the function f?
d) Come up with some more general notions of periodicity for continuous functions f on the interval $(0;\infty)$, which are related to the periodicity of sequences $(x_n) \in L(f)$. (For example, a function f is called *almost periodic*, if for any $\epsilon > 0$, there exists such $T > 0$ that $|f(x+T) - f(x)| < \epsilon$ for all sufficiently large x.)

4. Let $f(x) = g(\sin x)$, where g is some function continuous on the closed interval $[-1;1]$.

a) Do there exist such polynomial functions $g(x)$, that any sequence $(x_n) \in L(f)$ satisfies the following condition (*): the sequence $(x_1 + x_2 + \cdots + x_n)$, $n \in \mathbb{N}$ is bounded? Can the degree of such polynomial be any positive integer? Can such polynomial have all coefficients not equal to zero?
b) Do there exist polynomial functions $g(x)$ such that (**): there exists a constant $c = c(g)$, such that for any $(x_n) \in L(f)$ and for any positive integer n we have $|x_1+x_2+\cdots+x_n| < c(g)$?
c) Investigate if the sequences $(x_n) \in L(f)$ satisfy conditions (*) and (**), if $g(x) = \arcsin x$ or $g(x) = \sin x$.
d) Investigate when conditions (*) and (**) hold for other classes of functions $g(x)$.

5. Investigate the above questions for other classes of functions f.

10. **Pressing Coloured Buttons**

Let n and a_1, a_2, …, a_n be positive integers. Consider a collection of buttons with a_k of them in color k, where $k = 1, 2, \ldots, n$. Our main aim is to consider configurations of pressing *all buttons* under certain conditions.

Pressing several buttons at a time, if allowed, counts as a single *step*. Pressing the same button twice is not allowed. We can only press unpressed ones. Buttons of the same color are considered to be identical. In other words, it only matters how many buttons of the same color are pressed.

1.

a) Suppose that we can only press at least $t = 2$ buttons at a time, no two of which have the same colour. Find, with proof, a necessary and sufficient condition on a_1, a_2, …, a_n, for which it is possible to get all buttons pressed.
b) The above question for $3 \le t \le n$.
c) When the condition holds, find (or give bounds) in terms of t and the a_i-s the minimal number of steps required to do this process.

2. Now suppose $a_1 = a_2 = \ldots = a_n = 2$ and at a single step we can only press at least t buttons, with no two of the same color. For now let $t = 1$. Denote by A_n the number of possible ways to get all buttons pressed and by $A_{n,p}$ the number of possible ways to do that with exactly p steps in total (so that $A_n = \sum_{p=2}^{2n} A_{n,p}$).

a) Establish recurrence relations and give an explicit formula for $A_{n,p}$.
b) Find an asymptotic formula for A_n as $n \to \infty$.

c) Investigate the sums $\sum_{p=2}^{2n} A_{n,p} x^p$ for fixed n and $\sum_{n=p}^{\infty} A_{n,p} y^n$ for fixed p, in terms of $x, y \in \mathbb{R}$.
d) What happens in a), b), c), when $a_1 = a_2 = \cdots = a_n > 2$?
e) What happens in a), b), c), if $t = 2$ instead?

3. Again, let $a_1 = a_2 = \ldots = a_n = 2$, $t = 1$ at a single step we can press only buttons of different colors. Denote by S the set of all possible ways to press all buttons and choose an element of S uniformly. Study the following quantities: (find an explicit expression, behavior as $n \to \infty$, etc.)

a) The expected number of total steps performed.
b) The probability that exactly b buttons are pressed on the first step, when b is fixed. Generalize to b_1 buttons pressed on the first step, b_2 on the second, etc. b_l on the l^{th}, for fixed l and b_1, b_2, …, b_l.
c) The expected number of pressed buttons on the first step.

4. Let q be a prime number and $A_{n,p}$ be as in question 2.

a) When does q divide $A_{n,3}$? What about $A_{n,4}$?
b) Find upper and lower bounds on the largest power of q dividing $A_{n,p}$.

5. Consider the above problems for other a_1, a_2, …, a_n, t. Suggest and study additional directions of research.

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