PROBLEMS FOR THE 9^{th} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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Notation

$\mathbb{N} = \{1, 2, 3, \ldots\}$	set of natural numbers (positive integers)
$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$	ring of residue classes modulo n
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	sets of integer, rational, real and complex numbers
$\mathbb{R}_+$	set of positive real numbers
$\mathbb{F}_q$	finite field of size $q = p^k$
$M \otimes N$	mean-product of two functions M and N
$\gcd(x_1, \ldots, x_n)$	greatest common divisor of $x_1, \ldots, x_n$
$\mathbb{P}[.]$ and $\mathbb{E}[.]$	probability and expectation
$\det(A)$ or $\lvert A\rvert$	determinant of a matrix A
$\lvert x\rvert$	absolute value of x
$\#M$ or $\lvert M\rvert$	cardinality of a set M
$\sup\limits_{x \in M} f(x)$	supremum of a set $\{f(x) \mid x \in M\}$

1. Numbers from Strings

1. Four fours:

a) What is the largest number we can form with 4 fours by using only the four standard arithmetic operations? (You are allowed to concatenate several fours to obtain, for example, 44.)

b) What is the smallest positive number?

c) How many numbers can be formed ?

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2. Five fives:

a) Investigate this question for 5 fives if the additional (fifth) operation of exponentiation (a^b) is allowed.
b) Try another "fifth operation" instead.

3. In this question we consider the number of symbols in the expression we evaluate. For example, the string "$(4+4)*(4+4)$" uses 11 characters and evaluates to 64.

a) Using the digits 0 to 9, the 4 operations and brackets, what is the largest number using n symbols?
b) If we allow variables and assignment statements like "$a = 32, b = 77, a * b * b$", what is the largest number that can be obtained by evaluating the string with 10; 15 and 20 symbols? Try to generalise. We assume here that statements may involve only 4 arithmetic operations and possibly assignments. Different statements are separated by comma (which also counts as a symbol!), and the final result is the evaluated value of the last statement in the string.
c) If we allow one-argument function exp, what will be the result?

4. Define the "value" of a string of n symbols as $\log_d(\text{eval}(\text{string}))$, where $d = \#\{\text{string}\}$ is the number of *different* symbols used. What is the largest possible value of a string with $5, 10, 15$ symbols?

2. **Monotonous Functions**

Denote by $f(x), g(x), h(x)$ some real-valued functions defined for all $x \in \mathbb{R}$ such that:

$$f(x)g(x) = h(x) \quad \text{for all } x \in \mathbb{R}. \tag{1}$$

1. Do there exist strictly increasing functions $f(x)$ and $g(x)$ satisfying (1), if $h(x) = x$ is the identical function? Do such functions $f(x)$ and $g(x)$ exist, if in addition they are:

a) continuous for all $x \in \mathbb{R}$?
b) differentiable (twice differentiable, infinitely differentiable) for all $x \in \mathbb{R}$?
c) entire, that is represented as a sum of the power series converging for all $x \in \mathbb{R}$?
d) discontinuous at a fixed finite set of points on the line?
e) discontinuous at a fixed countable set of points on the line?

2. Give an answer to the above questions, if $h(x) = a\sin(x) + b\cos(x) + c$ is a trigonometric polynomial.

3. For each item in question 1, describe classes of functions $h(x)$ for which the answer is positive. In other words, describe all functions $h(x)$ that can be represented as (1) with additional conditions on the strictly increasing functions $f(x)$ and $g(x)$.

4. Investigate how non-monotonous the function $h(x)$ can be in each of the classes from the previous item. In particular, can the function $h(x)$ be non-monotonous for each $x \in \mathbb{R}$, that is the expression $(h(x) - h(a))(x - a)$ has different signs on any open interval containing x?

3. **Dense Analytic Curves**

The problem is concerned with the existence of *analytic* curves that are dense in $\mathbb{C}$.

1. Let L be any line in the complex plane:

$$L = \{at + b \,:\, t \in \mathbb{R}\}, \quad \text{where } a, b \in \mathbb{C}.$$

Describe all possibilities for $\exp(L)$. Recall that the complex exponential $\exp(z)$ is defined for $z = x + iy$ as:

$$\exp(z) = \sum_{n=0}^{\infty} z^n/n! = \exp(x)\exp(iy) = \exp(x)(\cos y + i \sin y).$$

2. Describe all possibilities for $\exp(\exp(L))$. In particular, state necessary and sufficient conditions for when this is dense in $\mathbb{C}$.

3. Show that $\exp(\exp(\exp(L)))$ is always dense in $\mathbb{C}$ unless a, b are exceptional. What are these exceptions?

4. Are there analogous results for maps of the form $\cos(z)$, or $\exp(z) + c$, or $a\cos(z) + bsin(z)$, or some other map?

5. Can you find similar results in three or more dimensions?

4. **Partitions of Regular Polygons**

1. Let $n \geq 1$ be an integer. We consider the following equation, where n, a, b and c are positive integers:

$$a^n + b^n = n^c.$$

a) Solve the equation for $n = 2$.
b) Solve the equation when n is a prime number.
c) Study the solutions of the equation when n is not prime.

2. Let $n \geq 3$ be an integer and let P_n be the regular polygon with n sides inscribed in the unit circle and whose vertices are the n complex roots of unity. Fix an integer $k \geq 3$. A k-angulation of P_n is a collection of diagonals of P_n that may intersect only at their endpoints and such that every face has degree k (a degree of a face is the number of diagonals or sides of P_n that surround the face). For $k \geq 3$, let $Z_n^{(k)}$ be the number of k-angulations of P_n.

a) Find a formula for $Z_n^{(3)}$. How does $Z_n^{(3)}$ behave for large n?
b) For $k \geq 4$, express $Z_n^{(3)}$. How does $Z_n^{(3)}$ behave for large n?

3. Let $\mathcal{A}_n^{(k)}$ be a k-angulation of P_n chosen randomly among all k-angulations of P_n. Let $\chi(\mathcal{A}_n^{(k)})$ be the length of the longest diagonal of $\mathcal{A}_n^{(k)}$. Find $a; b > 0$ such that:

$$\mathbb{P}(\chi(\mathcal{A}_n^{(3)}) > a) \to 1; \quad \mathbb{P}(\chi(\mathcal{A}_n^{(3)}) < b) \to 1,$$

as n tends to infinity.

4. Does there exist a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $0 \leq u \leq v$ we have:

$$\mathbb{P}(u \leq \chi(\mathcal{A}_n^{(3)}) \leq v) \to \int_u^v f(x)dx$$

as n tens to infinity?

5. Investigate questions 3 and 4 when $\mathcal{A}_n^{(3)}$ is replaced by $\chi(\mathcal{A}_n^{(k)})$.

6. Investigate question 3, 4 and 5, when $\chi(\mathcal{A}_n^{(k)})$ is replaced by the largest area of all faces in $\mathcal{A}_n^{(k)}$.

5. **Coin Toss Function**

Let $p \in (0;1)$ be a real number. Consider an infinite process of tossing a coin, such that on each trial the probability for heads is p. Form the random binary number $X = 0, a_1 a_2 a_3 \dots$ as follows: if the i-th toss gives heads, then $a_i = 1$, else $a_i = 0$. If $x \in [0;1]$ is a real number, then denote by $F_p(x)$ the probability that $X \leq x$.

1. Calculate $F_p((0, 11111100001)_2)$ and $F_p(1/2017)$ in terms of p. Can you give general algorithm(s) for calculating $F_p(x)$ for a given decimal x, if x is:

a) of the form $a/2^k$, $a; k \in \mathbb{N}$;
b) rational;
c) irrational.

2. Let n be a positive integer. Calculate the expected value of X^n.

3. Investigate the function $F_p(X)$ in more detail:

a) Show that $F_p(x)$ is strictly increasing in $[0;1]$.
b) Show that $F_p(x)$ is continuous in $[0;1]$.
c) For which p the function $F_p(x)$ uniformly continuous in $[0;1]$?

4. Let n be a positive integer. Find:

a) $\int_0^1 F_p(x)^n \, dx$.
b) the arc length of the graph of $F_p(x)^n$ on the interval $[0;1]$.

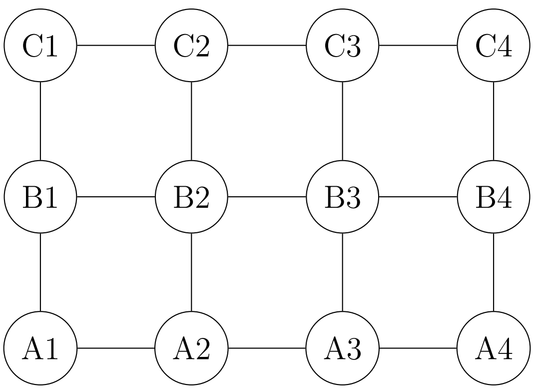
5. Let $p \neq 1/2$. Denote by $f_p(x)$ the derivative of $F_p(x)$ (at points x where it exists).

a) Show that $f_p(x)$ does not exist if x is of the form $a/2^k$, $a; k \in \mathbb{N}$.
b) For a fixed p, describe the behaviour of the derivative $f_p(x)$: for which $x \in \mathbb{R}$ does it exist (or is infinite) and at which points it does not exist? How to find its value?
c) Find all $x \in \mathbb{R}$ in which the derivative exists (and is not infinite) for all p.

6. Suggest and investigate generalisations. For example, one can define a similar function in k-number system, if instead of 2 outcomes (0 and 1) we have k outcomes ($0, 1, \dots, k-1$), with probabilities $p_0, p_1, \dots, p_{k-1}$.

6. **Rooks on Graphs**

1. Consider an m by n grid:



a) Place a rook on the lower left corner of this grid. In how many ways can the rook reach the upper right corner if in a single step it may move one edge up or one edge to the right, but not down or left? For example, the rock can move $A1-A2-B2-B3-B4-C4$ in the grid above.

b) Now suppose the rook can move in each of the four directions. In how many ways it can reach the upper left corner ($A1$ and $C1$ in the grid in the picture), if at the beginning it is placed on the lower left corner and the rook can't go through an edge more than once? An example of a valid path is $A1-A2-A3-A4-B4-C4-C3-C2-B2-B1-C1$ in the grid above. Start with small $m = 2, 3, 4, \ldots$

c) Find the number of rook paths, which do not pass through the same edge more than once and join the opposite vertices of the grid ($A1$ and $C4$ in the grid on the picture). Start with the case when the grid is square ($m = n$) and some small value of m.

2. Now assume the grid is infinite.

a) Find the number of paths which start and end at the same vertex, do not pass through the same edge more than once and go through exactly t edges (vertices do not need to be necessarily distinct).

b) Start at an arbitrary vertex and start moving along the grid with equal probability in each direction. Find the probability of the event that the path of the rook forms a cycle of length t without repeating edges.

3. Now instead of a grid we will generalise the problem by looking at a graph $G(V;E)$ without loops or multiple edges.

a) Find the number of cycles of length t in a complete graph $G = K_n$ with n vertices (the vertices can be repeated but the edges can not).

b) For which n does there exist a cycle which covers all edges of the complete graph K_n? (Each edge is used exactly once.)

4. Now remove one edge from the complete graph K_n.

a) Find the the number of cycles of length t in the new graph K'_n.

b) How the answer will change if we remove 2 edges from the complete graph K_n? Does the answer depend on which 2 edges were removed?

5. Suppose again that $G = K_n$ is the complete graph with n vertices. Start at an arbitrary vertex and start moving along the graph with equal probability. Give upper and lower bound for the probability of the event that the path forms a cycle of length t without repeating edges.

7. Weighted Sums of Distances

Let $\triangle ABC$ be a triangle and (λ, μ, ν) be a triple of real numbers with $\lambda + \mu + \nu \neq 0$. Let also X be a point in the plane of the triangle. Denote by d_1, d_2, d_3 the oriented distances from X to the lines BC, CA and AB, respectively. We are interested in the function:

$$f(X) = \lambda d_1^2 + \mu d_2^2 + \nu d_3^2$$

and in particular, in its minimum and maximum (if they exist).

1. Let $\lambda = \mu = \nu = 1$. Prove that the minimum of f exists and is attained only at the Lemoine point of $\triangle ABC$.

2. Let λ, μ, ν be non-negative.

a) Suppose $\lambda\mu\nu \neq 0$. Prove that $f(X)$ always has a minimum and describe the points (in terms of λ, μ, ν) at which it is attained.

b) What happens in the case $\lambda\mu\nu = 0$?

3. Consider the case when exactly one of λ, μ or ν is negative.

4. Consider the general case:

a) Find all triples (λ, μ, ν), $\lambda + \mu + \nu \neq 0$ (and the respective attainable points X) for which $f(X)$ has a minimum, and those for which it has a maximum.
b) Answer the above question in the cases when $\lambda + \mu + \nu = 0$.

5. Answer question 4 for the function

$$f(X) = \lambda d_1^3 + \mu d_2^3 + \nu d_3^3.$$

6. Now instead of the triangle consider a tetrahedron and quadruples $(\lambda_1; \lambda_2; \lambda_3; \lambda_4)$ of real numbers with non-zero sum. Similarly, we can define a function

$$f(X) = \lambda_1 d_1^2 + \lambda_2 d_2^2 + \lambda_3 d_3^2 + \lambda_4 d_4^2,$$

where d_i are the distances from a point X in space to the planes containing the faces of the tetrahedron $A_1A_2A_3A_4$. Find all quadruples (and the respective attainable points X) for which $f(X)$ has a minimum, and those for which it has a maximum.

7. Similarly, consider an n-dimensional simplex and an analogously defined function (with squared distances) for points in n-dimensional space.

8. A Communication Network

Consider the communication network consisting of nodes and communication channels between some of the nodes. Two nodes can communicate either via the direct channel between them (if any), or via any other channels connecting these nodes. The network is called valid, if any two nodes are able to exchange messages.

There is some secret information split between nodes in such a way that each node contains a part of the secret not known to any other node in the network. Any node N can broadcast a message. In this case all its information (its initial part of the secret and all secret parts received from other nodes up to this moment) becomes known to all nodes connected with N via the direct channel. The minimal number of such messages in order for all nodes to obtain the entire secret (i.e. the secrets from all other nodes) is called *a secret connectivity number* or just a *secret number* of the network.

We can model such communication network by a graph G, where vertices of G correspond to nodes and edges of G become direct communication channels. A valid communication network corresponds to a connected simple graph G.

Denote by $s(G)$ the secret connectivity number of the corresponding network. For example, if $G = P_4$ is a simple chain with vertices $\{v_1, v_2, v_3, v_4\}$ and edges $\{v_1v_2, v_2v_3, v_3v_4\}$, then $s(G) = 5$, and one of the optimal sequences of messages is $\{v_1, v_4, v_3, v_2, v_3\}$.

1. Find $s(G)$ for the following graphs:

a) the complete graph K_n;
b) the simple cycle graph C_n (i.e. the graph represented by a regular n-gon);
c) the simple chain P_n;
d) the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$;
e) the graph of a three-dimensional cube;
f) the Petersen graph.

2. Let T be an arbitrary connected tree with n vertices. Find $s(T)$ and describe an algorithm generating an optimal sequence of messages.

3. Consider the case of a connected unicyclic graph, that is a connected graph with a single cycle.

4. Consider the graph $G_{m,n}$ of the m by n grid (see Problem 6).

5. Try to relate $s(G)$ or at least estimates for $s(G)$ with other known numeric invariants of graphs.

9. **An Approximation Problem**

1. Consider the following subset of rational numbers
$$U = \left\{ x - \frac{1}{x} \mid x \in \mathbb{Q},\, x \neq 0 \right\}.$$

a) Show that U is dense in the real line (that is for any non-empty interval $I \subset \mathbb{R}$, there exists an element $r \in U$ which belongs to I)?
b) Denote by aU the set of products ar with $r \in U$. Show that for any real $a \neq 0$, the set aU is also dense in $\mathbb{R}$.

2. Study intersections $aU \cap bU$ where a and b are integers. In particular, determine whether $aU \cap bU$ is a) empty; b) infinite; c) dense in $\mathbb{R}$.

3. Let $n > 2$ be a natural number. Study intersections $a_1U \cap \ldots \cap a_nU$ where $a_1, \ldots, a_n$ are integers.

4. More generally, let $f\colon D \to \mathbb{R}$ be a real-valued function defined on a domain $D \subset \mathbb{R}$. Consider the following set
$$U_f = \{ f(x) \mid x \in \mathbb{Q} \cap D \}.$$
For example, in the previous questions $U = U_f$ with $f(x) = x - 1/x$ and $D = \mathbb{R} \setminus \{0\}$.

Check the density of sets $a_1U_f \cap \ldots \cap a_nU_f$ where n is a natural number and $a_1, \ldots, a_n$ are integers, in the following situations:

a) f is a polynomial with integer coefficients;
b) $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with integer coefficients (start with polynomials of degree at most 2);
c) f is a trigonometric function;
d) $f(x) = \frac{S(x)}{T(x)}$, where $S(x)$ and $T(x)$ are trigonometric polynomials with integer coefficients.

5. Suggest and investigate additional directions of research.

10. **Kinds of Convexity**

Recall that a function $f(x)$ defined on an interval $I \subset \mathbb{R}$ is called *convex* on I, if for any $x, y \in I$ and any $\lambda \in [0; 1]$ we have the following inequality:
$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

1. Let $f(x)$ be a convex function on I. Let $a_1 \leq a_2 \leq \ldots a_n$, $b_1 \leq b_2 \leq \ldots b_n$ and $a_i + b_j \in I$ for all $1 \leq i, j \leq n$. Prove the following inequality:
$$\sum_{i=1}^{n} f(a_i + b_i) \geq \sum_{i=1}^{n} f(a_i + b_{n-i+1}).$$

2. Let $f(x)$ be a convex function on I. Let $a_1 \leq a_2 \leq \ldots a_n$, $b_1 \leq b_2 \leq \ldots b_n$ and $a_ib_j \in I$ for all $1 \leq i, j \leq n$.

a) Is it true that
$$\sum_{i=1}^{n} f(a_ib_i) \geq \sum_{i=1}^{n} f(a_ib_{n-i+1})?$$

b) If not, try to find additional conditions on a_i and b_i, so that it becomes true.

3. Fix a positive real number s. Let us say that the function $f(x)$ is s-convex of first kind, if for any $x, y \in I$ and any $\lambda \in [0; 1]$ we have the following inequality:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x)^s + (1-\lambda) f(y)^s.$$

a) For which s there exist a function $f(x)$ on a closed interval $I \subset \mathbb{R}$, which is s-convex of first kind, but not convex?
b) Which of the inequalities from questions 1 and 2 hold for any s-convex function of first kind?

4. Let us say that the function $f(x)$ is s-convex of second kind, if for any $x, y \in I$ and any $\lambda \in [0; 1]$ we have the following inequality:

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y).$$

Solve question 3 for s-convex functions of second kind.

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