PROBLEMS FOR THE 5^{th} INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

5TH – 12TH JULY 2013, IAŞI (ROMANIA)

Contents

Keywords: 1. combinatorics – 2. differential geometry – 3. dynamical systems – 4. optimal stopping theory, probability – 5. plane geometry – 6. additive number theory – 7. graph theory, colouring – 8. games – 9. number theory – 10. combinatorics – 11. analysis, iterations.

Notation

$\mathbb{N} = \{1, 2, 3, \dots\}$	set of natural numbers (positive integers)
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$	sets of integer, rational and complex numbers
$\mathbb{R}$, $\mathbb{R}^2$	real line and real plane
$\gcd(x_1, \dots, x_n)$	greatest common divisor of $x_1, \dots, x_n$
$\operatorname{lcm}(x_1, \dots, x_n)$	least common multiple of $x_1, \dots, x_n$
$[a, b]$, $]a, b[$	closed and open intervals in $\mathbb{R}$
$f^k(x)$	k^{th} iteration of a function f at a point x
$C(\mathbb{R})$	set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$
$\lvert XY \rvert$	length of a segment XY

1. Pentominoes

A *pentomino* is a flat figure composed of five unit squares, connected along their edges.

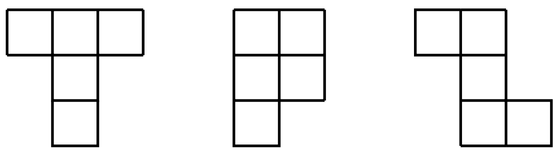
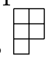
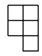
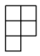
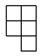


Figure 1. The T, P and Z pentominoes.

Date: March 26, 2013.

1. We say that two pentominoes are of the same *type* if they are same up to a rotation or up to a symmetry. Classify the pentominoes. For example, the pentominoes  and  are of the same type (they are the same up to a symmetry, but not up to a rotation).

2. Determine whether it is possible to tile the real plane by a given type of pentominoes.

3. Consider the P pentominoes, that is  and  together with their rotations.

a) For which $n \in \mathbb{N}$ is it possible to tile a $5 \times n$ rectangle (to cover it without overlap and without gaps)?

b) Find all $m, n \in \mathbb{N}$ for which one can tile an $m \times n$ rectangle.

c) In general, given positive integers m and n, what is the maximal number of P pentominoes that can be placed into an $m \times n$ rectangle along grid lines and without overlap?

4. The same problem for other types of pentominoes.

2. **Steady Triangulations**

Let M be a two-dimensional surface, possibly with boundary. Denote by $\Delta(M)$ a *triangulation* of the surface, that is a subdivision of M into triangles with no overlap and no gaps. Those vertices of the triangulation which do not lie on the boundary of M are called *inner vertices*. An example of a triangulation is given in Figure 2, where the surface M is a regular hexagon, and $\Delta(M)$ consists of 19 triangles and has 8 inner vertices.

We will say that a triangulation $\Delta(M)$ is *steady* if the following two conditions are satisfied:

i) for any two triangles ABC and ABD of $\Delta(M)$ sharing a common edge, one has

$$\frac{|AC|}{|AD|} = \frac{|BC|}{|BD|},$$

ii) each inner vertex of $\Delta(M)$ is a vertex of exactly six triangles of $\Delta(M)$.

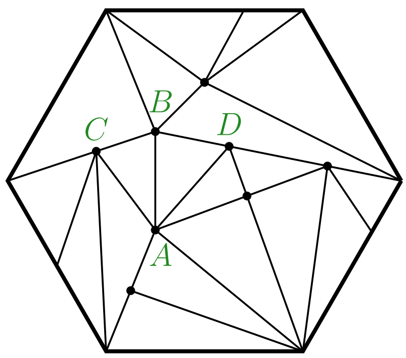


FIGURE 2. A non-steady triangulation of a regular hexagon.

1. Consider a triangulation of a convex hexagon satisfying only condition ii). How many inner vertices can it have?

2. Describe all steady triangulations of a regular hexagon.

3. Let n and k be positive integers. Suppose that M is a regular n-gon, and that there exists a steady triangulation $\Delta(M)$ with exactly k inner vertices. Is it true that all triangles of $\Delta(M)$ must be regular? Start with $n = 3, 4, 5$ and small k.

4. Let M be a sphere. Define straight lines, length of a segment and angles between two lines on the sphere. Investigate properties of steady triangulations of M.

5. Let M be the real plane $\mathbb{R}^2$. Must all triangles of a steady triangulation $\Delta(M)$ be regular?

6. Consider other surfaces.

3. **A Snooker**

Two balls coloured white and red are laid on a billiard table with no pockets. Ronnie wants to hit the red ball by the white one. However, Mark is going to put a finite number of black balls on the table, and his goal is to prevent Ronnie from hitting the red ball without previously touching a black one. Can Mark succeed? If yes, how many black balls will he need?

We suppose that the balls are infinitesimal (like points), the motion is a straight line and with constant speed, the reflections from the boundary are specular, and the white ball is not allowed to hit a corner. Finally, Mark may also put black balls on the boundary of the table. Consider the cases that the billiard table has the following form:

1. A square.

2. A right-angled triangle.

3. An L-billiard table (see Figure 3).

4. A regular n-gon, where $n \in \mathbb{N}$.

5. A polyomino: a collection of unit squares connected along their edges (as in Figure 3).

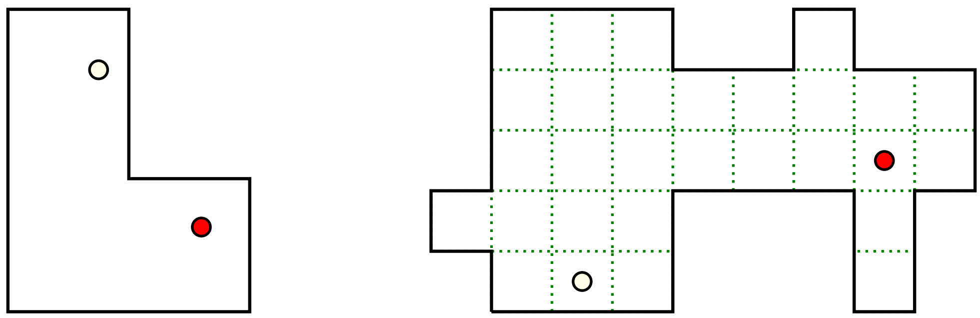


FIGURE 3. An L-billiard table and a billiard table in the form of a polyomino.

6. A convex polygon whose angles are rational multiples of π.

7. A convex polygon whose angles are irrational multiples of π.

8. Give examples of non-convex polygonal tables such that the answer (yes or no) depends on the initial position of the red and the white balls, and both answers occur.

4. **Best Choice Problem**

Fix an integer $n \geq 1$. Alice and Bob play the following game.

A closed opaque box contains n balls. On these balls are written n *distinct* rational numbers, with exactly one number written on each ball. Alice's goal is to find the ball with the greatest number. Bob takes out one ball from the box in a uniformly random manner (this means that the probability of choosing a particular ball is $1/n$) and shows Alice its number. Alice must either keep this ball, or reject it. If Alice keeps the ball, then the game is finished. If Alice rejects the ball, then this ball is removed from the game. Then Bob takes out a new ball randomly out of the box, and the previous procedure is repeated. Finally, if there is only one ball left in the box, Alice has to keep it.

At each moment, Alice only knows how many balls remain in the box and the numbers on the balls already rejected. Alice wins the game if she accepts the ball with the greatest number, otherwise Bob wins.

This game is known in the literature as the *Best Choice Problem* or the *Secretary Problem.*

1. Find a strategy that allows Alice to win with the maximal probability P_{max} and determine the value of P_{max}. Start with $n = 3, 4$.

2. Let $m \geq 2$ be an integer. A ball with number q is called *m-good* if there exist at most $[n/m]$ balls whose numbers are greater than q. Here $[x]$ is the integer k such that $k \leq x < k + 1$.

Now, Alice and Bob play a similar game, in which the only change is that Alice wins if she keeps any m-good ball, and loses otherwise.

a) For $m = 2$, find a strategy that allows Alice to win with the maximal probability Q_{max} and determine the value of Q_{max}.

b) For $m > 2$, find a strategy that allows Alice to win with the maximal probability Q_{max} and determine the value of Q_{max}.

3. Suggest and study additional directions of research.

5. **Congruent Polygons**

1. Prove the following statements, or find counterexamples:

a) If all 5 sides of a convex pentagon have the same length and 3 diagonals have the same length, then the pentagon must be regular.

b) If 4 sides of a convex pentagon have the same length and 4 diagonals have the same length, then the pentagon must be regular.

c) If 3 sides of a convex pentagon have the same length and 5 diagonals have the same length, then the pentagon must be regular.

d) If all 5 sides of a convex pentagon have the same length and 2 diagonals have the same length, then the pentagon must be regular.

2. Let $n \geq 3$ be a positive integer. Find the set of pairs of positive integers s and d such that the following statement is always true:

If s sides of a convex n-gon have the same length and d diagonals have the same length, then the polygon must be regular.

3. Let $n \geq 3$ be a positive integer. Find the set of pairs of positive integers s and d such that the following statement is always true:

Let $A = A_1 \ldots A_n$ be a convex polygon, and let $B = B_1 \ldots B_n$ be a regular polygon. If s sides in A have the same length as corresponding sides in B and d diagonals in A have the same length as corresponding diagonals in B (we say that A_iA_j *corresponds* to B_iB_j), then the polygons are congruent.

4. Prove the following statements, or find counterexamples:

a) If the 5 sides and 3 diagonals in a convex pentagon have the same length as corresponding sides and diagonals in another convex pentagon, then the pentagons must be congruent.

b) If 4 sides and 4 diagonals in a convex pentagon have the same length as corresponding sides and diagonals in another convex pentagon, then the pentagons must be congruent.

5. Fix a natural number $n \geq 3$. Find all pairs of positive integers s and d such that the following is always true:

Let $A = A_1 \ldots A_n$ and $B = B_1 \ldots B_n$ be two convex polygons. If s sides and d diagonals in A have the same length as corresponding sides and diagonals in B (we say that A_iA_j *corresponds* to B_iB_j), then the polygons are congruent.

6. **An Expensive Dinner**

There are n friends, numbered 1 through n. They are all going to a restaurant for dinner tonight. They are all very good at math, but they are all very strange: the a^{th} friend will be unhappy unless the total cost of the meal is a positive integer, and is divisible by a.

The friends enter the restaurant one at a time. As soon as someone enters the restaurant, if that person is unhappy then the group will call a waiter immediately.

As long as there is at least one unhappy person in the restaurant, one of those unhappy people will buy the lowest-cost item that will make him or her happy. This will continue until nobody in the restaurant is unhappy, and then the waiter will leave. Fortunately, the restaurant sells enough food at every integer price.

The friends could choose to enter the restaurant in any order. After the waiter has been called, if there is more than one unhappy person in the restaurant, any one of those unhappy people could choose to buy something first. The way in which all of those choices are made could have an effect on how many times the group calls a waiter.

For example, if $n = 3$ and the friends arrive in the order $[1, 2, 3]$, then a waiter will be called three times. (#1 arrives, is unhappy, calls a waiter and buys something costing 1 euro. #2 arrives next, is unhappy, calls a waiter and buys something costing 1 euro. Now nobody is unhappy. #3 arrives next, is unhappy, calls a waiter and buys something costing 1 euro. Now #2 is unhappy and buys something costing 1 euro. Now #3 is unhappy and buys something costing 2 euro, for a total of 6 euros. Finally nobody is unhappy, and a waiter was called three times.) However, if the friends arrive in the order $[3, 1, 2]$, then a waiter will be called 2 times.

1. Denote by W_{max} the maximal number of times the friends might call a waiter. Find W_{max} or estimate it.

2. Denote by W_{min} the minimal number of times the friends might call a waiter. Find W_{min} or estimate it.

3. Suppose now that only friends with numbers $a_1, \ldots, a_k$ will come to the restaurant. Find or estimate W_{max} and W_{min} in this case. Here $a_1, \ldots, a_k$ are distinct positive integers.

4. Denote by W the number of times the friends will call a waiter. Determine the set of all possible values of W. Consider the following cases:

a) The friends are numbered 1 through n.

b) The friends have numbers $a_1, \ldots, a_k$ (distinct positive integers).

5. Denote by U the total number of times the friends were unhappy. Determine the set of all possible values of U in the cases a) and b) above.

6. Suggest and investigate additional directions of research.

7. **Map Colourings**

A *map* is a separation of the real plane into a finite number of regions whose boundaries are broken lines. Two such regions are said to be *adjacent* if their boundaries share a common segment which is not reduced to a point. A *colouring* of a map is the act of assigning a colour to each of its regions (adjacent regions may have the same colour).

Let $n \geq 2$ be a positive integer, and let C be a map. Define an *n-cycle* of C as a sequence $R_1, \ldots, R_n$ of distinct regions such that R_i and R_{i+1} are adjacent for each $1 \leq i \leq n-1$, as well as R_n and R_1. A colouring of the map C will be called *n-admissible* if it has no monochrome n-cycle, that is, if in any n-cycle there are at least two regions of different colours.

Denote by $K(n)$ the least positive integer such that any map C has an n-admissible colouring with at most $K(n)$ colours. For example, due to a theorem of Kenneth Appel, Wolfgang Haken

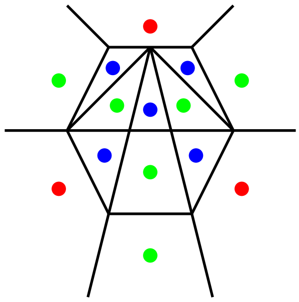


FIGURE 4. A colouring of a map.

and John A. Koch, any map can be coloured using at most four colours so that no two adjacent regions have the same colour, which implies that $K(2) \leq 4$.

1. Find the exact value of $K(3)$, or give lower and upper bounds.

2. The same problem for $K(n)$ with $n \geq 4$.

3. Find relations between the numbers $K(n)$. For instance, is it true that $K(n+1) \leq K(n)$ for any $n \geq 2$?

4. Does there exist an integer $n \geq 2$ such that $K(n) = 2$?

5. Find the least number K_∞ such that, for any map C, there exists a colouring of C using at most K_∞ colours which is n-admissible for all $n \geq 3$ simultaneously.

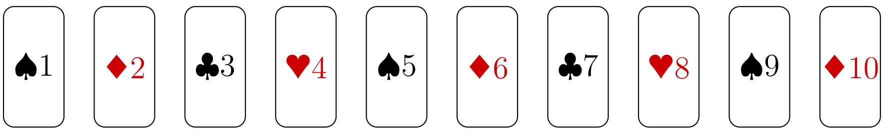
8. **Black Jack**

1. Ten cards with numbers from 1 to 10 are laid face-up on the table. Jane picks a card. Then Joe chooses several cards one after another until his total of chosen cards exceeds (is strictly greater than) the card chosen by Jane. Then Jane chooses until her cumulative total exceeds that of Joe, and so on. The first player to get

a) 21 exactly,

b) 21 or more

wins. Which of the players has a winning strategy?



2. Let n and K be positive integers such that $n < K$. Consider the same game with cards numbered 1 through n. The winner is the first player to get a) K exactly, b) K or more.

3. In general, suppose that there are n cards with integer numbers $a_1 < a_2 < \ldots < a_n$, and let K be a natural number. Jane and Joe play the same game trying to get a pile with the sum a) K exactly, b) K or more. Determine the set of increasing sequences $(a_i)_{1 \leq i \leq n}$ for which Joe has a winning strategy. Consider also the case that the numbers on some of the cards are negative.

4. Investigate the game for non-decreasing sequences $a_1 \leq a_2 \leq \ldots \leq a_n$, when there are cards with equal numbers. For example, take a standard deck of 52 cards with the following rank:

Ace=1 2 3 4 5 6 7 8 9 10 Jack=11 Queen=12 King=13.

5. Suggest and investigate related questions.

9. **Prime Quadruplets**

Two positive integers p and q are called *prime twins* if they are both prime and $q = p + 2$. For example, $(3, 5)$, $(5, 7)$, $(11, 13)$, $(17, 19)$ are pairs of prime twins. If (p_1, q_1) and (p_2, q_2) are two pairs of prime twins, then we will say that (p_1, q_1, p_2, q_2) is a *prime quadruplet*.

1. Find all natural n for which there exist prime twins p and q such that the following two numbers are also prime:

a) $2^n + p$ and $2^n + q$;

b) $2^{2n} + 2013 \cdot 2^n + 2014 + p$ and $2^{2n} + 2013 \cdot 2^n + 2014 + q$;

c) $9^n + 7^n + 5^n + 3^n + p$ and $9^n + 7^n + 5^n + 3^n + q$.

2. Let $a > 1$ be a positive integer. Are there only finitely many natural n for which there exists a prime quadruplet $(p, q, a^n + p, a^n + q)$?

3. Let $a > 1$ be a positive integer, and let $P(x)$ be a non-constant polynomial with integer coefficients. Are there finitely many $n \in \mathbb{N}$ such that the quadruplet $(p, q, P(a^n)+p, P(a^n)+q)$ is prime for some prime twins p and q?

4. Let $a_1, \ldots, a_k$ be positive integers greater than 1. Estimate the number of natural n for which there exist prime twins p and q such that

$$a_1^n + \cdots + a_k^n + p \quad \text{and} \quad a_1^n + \cdots + a_k^n + q$$

are also prime twins.

10. **Puzzles**

1. A student took a sheet of paper in the form of an $n \times n$ square grid and cut it into k pieces along grid lines. It turned out that there is only one way to assemble the $n \times n$ square back (the uniqueness is up to a rotation around the center of the square). The pieces are allowed to be rotated, but cannot be turned over.

Find the maximal value of k or give an upper bound. Determine the set of all possible values of k. Distinguish the following cases:

a) Some pieces may have the same shape, and interchanging them does not give a new assemblage.

b) The pieces must be pairwise distinct.

2. The same problem for a sheet of paper in the form of an $n \times n$ equilateral triangular grid.

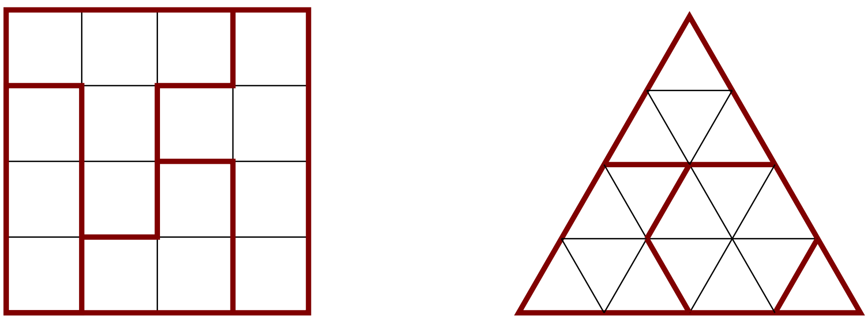


FIGURE 5. A 4×4 square grid and a 4×4 equilateral triangular grid (with puzzles).

3. Study the problem for other planar shapes, other types of grids, etc.

4. Formulate and investigate 3-dimensional analogs.

11. **Dense Iterations**

A set M of real numbers is called *dense* in $\mathbb{R}$ if any non-empty open interval $]a, b[$ of the real line contains at least one element of M. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We will say that f is *spreading at a point* $p \in \mathbb{R}$ if the set of iterations

$$\{f(p), f^2(p), \ldots, f^k(p), \ldots\}$$

is dense in $\mathbb{R}$, where $f^k(p) = \underbrace{f(f(\ldots f}_{k \text{ times}}(p)\ldots))$ denotes the k^{th} iteration of f at the point p.

1. Does there exist a real polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$ of degree n which is spreading at 0? Start with $n = 1, 2, 3$.

2. Denote by $C(\mathbb{R})$ the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and by $S_p(\mathbb{R})$ the subset of those continuous functions which are spreading at a point p. Study properties of such subsets. For instance, is $S_p(\mathbb{R})$ infinite, uncountable?

3. Is there a continuous function which is spreading at only one point? Describe the intersection of two subsets $S_p(\mathbb{R})$ and $S_q(\mathbb{R})$ for some points $p, q \in \mathbb{R}$.

4. Formulate and investigate an analogous problem for the interval $[0, 1]$ instead of $\mathbb{R}$ and the functions $f : [0, 1] \to [0, 1]$.

5. Suggest and study additional directions of research.

E-mail address: `organizers.itym@gmail.com`
URL: `http://www.itym.org/`