

# PROBLEMS FOR THE 1<sup>st</sup> INTERNATIONAL TOURNAMENT OF YOUNG MATHEMATICIANS

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## 1. Specular Colourings

1. What is the maximum number of cells of an  $m \times n$  grid that can be coloured blue, such that no two blue cells are symmetric with respect to any horizontal or vertical line of the grid?

2. Some cells of an  $m \times n$  grid are coloured blue. We call such a colouring *specular* if for any interior horizontal or vertical line of the grid there are two blue cells that are symmetric with respect to this line. Denote by  $S(m, n)$  the minimal number of blue cells in a specular colouring of an  $m \times n$  grid.

Find  $S(m, n)$  or estimate it (give lower and upper bounds).

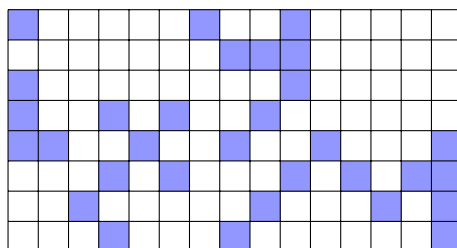


FIGURE 1. An example of specular, but not minimal, colouring of an  $8 \times 15$  grid.

3. Formulate and investigate 3-dimensional analogs of the problem.

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## 2. A Functional Equation

Let  $k$  be a constant real number.

1. Find some (all) functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  with the property that  $f(f(x) + kx) = xf(x)$  for all real numbers  $x$ .

2. Find all solutions  $f : \mathbb{R} \rightarrow \mathbb{R}$  of the functional equation

$$(1) \quad f(f(x) + f(y) + kxy) = xf(y) + yf(x), \quad x, y \in \mathbb{R}.$$

Consider the case when  $f$  is (a) a polynomial, (b) a continuous function, (c) an arbitrary function.

3. Let  $n > 2$  be a positive integer. Find some (all) functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(f(x_1) + f(x_2) + \dots + f(x_n) + kx_1x_2\dots x_n) = x_1f(x_2) + x_2f(x_3) + \dots + x_nf(x_1)$$

for all  $x_1, x_2, \dots, x_n \in \mathbb{R}$ .

4. Suggest and investigate other generalisations of the functional equation (1).

## 3. Monotonic Squares

We say that a positive integer  $a = a_k a_{k-1} \dots a_1 a_0$ , where  $0 \leq a_i \leq 9$  are the digits of  $a$  in base 10, is an *increasing square* if  $a = b^2$  for some integer  $b$  and  $a_k \leq a_{k-1} \leq \dots \leq a_1 \leq a_0$ . For instance,  $13456 = 116^2$ .

If we have the reverse inequalities  $a_k \geq a_{k-1} \geq \dots \geq a_1 \geq a_0$ , then the square  $a$  is called *decreasing*. For instance,  $8874441 = 2979^2$ .

Let  $a = b^2$  and  $c = d^2$  be two squares with the following representations in base 10:

$$a = a_k a_{k-1} \dots a_1 a_0, \quad b = b_l b_{l-1} \dots b_1 b_0 \quad \text{and} \quad c = c_m c_{m-1} \dots c_1 c_0, \quad d = d_n d_{n-1} \dots d_1 d_0.$$

We say that a pair of squares  $a$  and  $c$  is *ordered*, and write  $a \prec c$ , if the sequence  $a_0, a_1, \dots, a_{k-1}, a_k$  is a subsequence of  $c_0, c_1, \dots, c_{m-1}, c_m$  and the sequence  $b_0, b_1, \dots, b_{l-1}, b_l$  is a subsequence of  $d_0, d_1, \dots, d_{n-1}, d_n$ . For instance,  $1156 = 34^2 \prec 111556 = 334^2$ .

A set of squares  $F$  is called a *family* if any pair of squares from  $F$  is ordered.

1. Find infinite families of increasing squares. For instance,  $1156 = 34^2 \prec 111556 = 334^2 \prec 11115556 = 3334^2 \prec \dots$

2. Is there any infinite family of decreasing squares?

3. How many elements are in a maximal increasing family? For example, can it have exactly (a) one increasing square? (b) two increasing squares? (A *maximal increasing family*  $F$  is a family of increasing squares, such that any increasing square  $a$  with the property “either  $a \prec c$  or  $c \prec a$  for all  $c \in F$ ” is already in  $F$ .)

3. How many elements can a maximal family of decreasing squares have?

4. Investigate the problem in other bases.

## 4. Minimality of Inscribed Polygons

We say that a polygon  $P$  is *inscribed* in a polygon  $Q$  if the vertices of  $P$  lie on edges of  $Q$ , no two on the same edge. (A vertex of an inscribed polygon  $P$  is not allowed to coincide with a vertex of the polygon  $Q$ .)

1. A triangle  $T$  is inscribed in a triangle  $ABC$ , so that  $ABC$  is divided into four triangles:  $T_1, T_2, T_3$  and  $T$  (see the picture).

- (a) Is it always true that  $area(T) \geq \min\{area(T_1), area(T_2), area(T_3)\}$ ?
- (b) Can  $T$  have a bisector (median, perimeter, angle, inscribed or circumscribed circle, etc.) smaller than all bisectors (medians, perimeters, angles, inscribed or circumscribed circles, etc.) of the triangles  $T_1, T_2, T_3$ ?

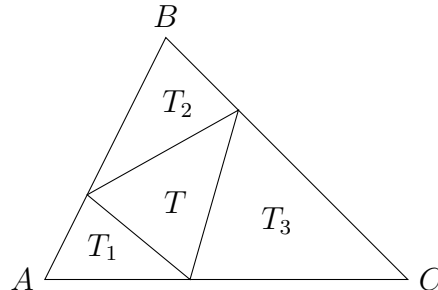


FIGURE 2. A triangle  $T$  inscribed in a triangle  $ABC$ .

2. A convex polygon  $P = P_1P_2 \dots P_m$  is inscribed in a convex polygon  $Q = Q_1Q_2 \dots Q_n$ , where  $3 \leq m \leq n$ , so that  $Q$  is divided into  $m + 1$  parts. Can the polygon  $P$  possess a minimality property compared to the parts (for instance, can  $P$  have the smallest area, perimeter, angle, diagonal, etc.)?

3. Formulate and investigate 3-dimensional analogs of the problem.

### 5. Integer-valued Polynomials

An *integer-valued polynomial*  $q(x)$  is a polynomial taking an integer value  $q(n)$  for every positive integer  $n$ . Denote by  $\mathbb{Q}_0[x]$  the set of all integer-valued polynomials with rational coefficients, that is

$$\mathbb{Q}_0[x] = \{q(x) \in \mathbb{Q}[x] \mid q(n) \in \mathbb{Z}, \forall n \in \mathbb{N}\}.$$

Let  $p$  be a prime number and let  $\mathbb{Z}_p = \{0, 1, \dots, p - 1\}$  be the set of residues modulo  $p$ .

- 1. Describe the set of integer-valued polynomials  $q(x) \in \mathbb{Q}_0[x]$  with the property  $q(n) \equiv 0 \pmod p$  for all  $n \in \mathbb{N}$ .
- 2. Let  $q(x)$  be a polynomial from  $\mathbb{Q}_0[x]$ . Define whether the sequence  $(q(n) \pmod p)_{n \in \mathbb{N}}$  is periodic and, if it is, find or estimate its period.
- 3. We say that a sequence  $(\alpha_n)_{n \in \mathbb{N}}$  of elements of  $\mathbb{Z}_p$  is *realisable* if there exists an integer-valued polynomial  $q(x) \in \mathbb{Q}_0[x]$  such that  $q(n) \equiv \alpha_n \pmod p$  for all  $n \in \mathbb{N}$ . Describe the set of realisable sequences.
- 4. Let  $(\alpha_n)_{n \in \mathbb{N}}$  be a realisable sequence. Describe the set of integer-valued polynomials  $q(x) \in \mathbb{Q}_0[x]$  such that  $q(n) \equiv \alpha_n \pmod p$  for all  $n \in \mathbb{N}$ .
- 5. Describe the set  $\mathbb{Q}_0[x]$ .

### 6. Pattern Graphs

Let  $n$  be a positive integer. A *pattern of length  $n$*  is a two-line table  $\begin{matrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{matrix}$ , where  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  are some rearrangements of the numbers  $1, 2, \dots, n$ .

Define two operations  $A$  and  $B$  on patterns as follows

$A$  : replace each number  $a$  of the first line with the number that is in the  $a$ 'th place (from left to right) of the second line,

$B$  : replace each number  $b$  of the second line with the number that is in the  $b$ 'th place (from left to right) of the first line.

We can construct an oriented labelled graph  $G_n$  whose vertices are all the patterns of length  $n$ , and such that for any two vertices  $v$  and  $w$  there is an  $A$ -arrow (resp. a  $B$ -arrow) from  $v$  to  $w$  if the pattern  $w$  is obtained from the pattern  $v$  by applying the operation  $A$  (resp. the operation  $B$ ).

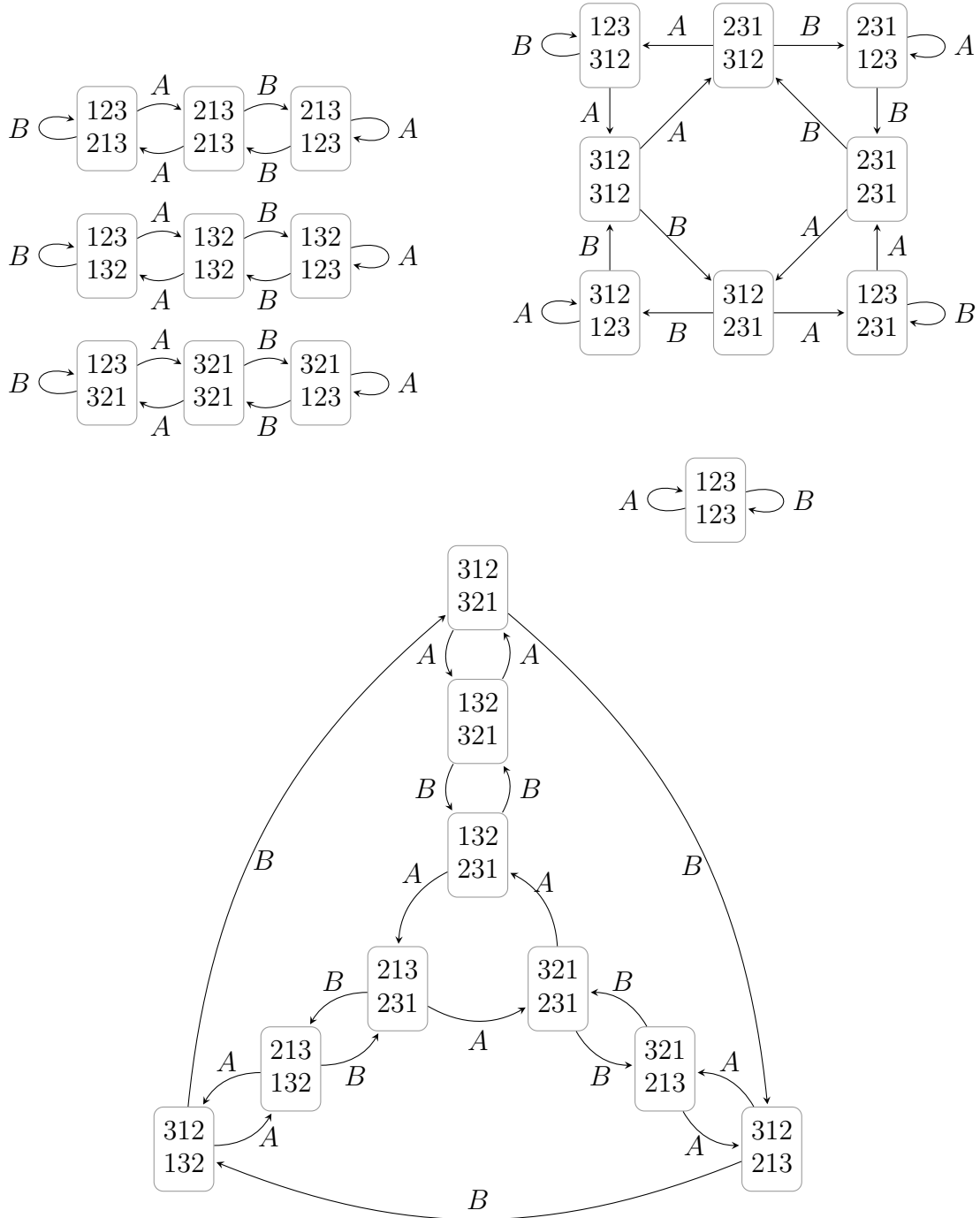



FIGURE 3. The graph  $G_3$  has 7 connected components (the 7th one with 9 vertices is not shown on the picture).

1. Can the pattern  $\begin{matrix} 2 & 1 & 3 & 4 & \dots & n-1 & n \\ 2 & 3 & 4 & 5 & \dots & n & 1 \end{matrix}$  be obtained from the pattern  $\begin{matrix} 2 & 3 & 1 & 4 & \dots & n-1 & n \\ 2 & 3 & 4 & 5 & \dots & n & 1 \end{matrix}$  using the operations  $A$  and  $B$ ?
2. How many connected components of the graph  $G_n$  have exactly (a) 2 patterns? (b) 3 patterns?
3. Denote by  $g_n$  the number of connected components of the graph  $G_n$ . Find  $g_n$  (give a formula) or estimate it (give lower and upper bounds).
4. Study geometric properties of the connected components of  $G_n$ . Can they be drawn nicely in the plane: without intersections of the edges, symmetrically, etc.?
5. As a generalisation one could consider three-line patterns with  $3 \cdot 2 = 6$  operations on them. Investigate this generalisation.

### 7. Placements of Pentominoes

1. Given an  $m \times n$  rectangle, denote by  $T(m, n)$  the minimum number of non-overlapping pentominoes  that must be placed (along the grid lines) so that there is no place on the free cells for another pentomino? Find or estimate the number  $T(m, n)$  and give an algorithm for constructing suitable placements.

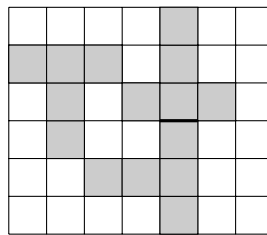



FIGURE 4. A placement of pentominoes on a  $6 \times 7$  rectangle:  $T(6, 7) = 3$ .

2. Two players alternately place pentominoes  on the free cells of an  $m \times n$  rectangle, along the grid lines. The loser is the one who cannot place a pentomino. Does any player have a winning strategy?
3. Study the previous questions for other polyominoes.

### 8. Positivity of Symmetric Polynomials

A polynomial  $P(x, y)$  with real coefficients is *symmetric* if the equality  $P(x, y) = P(y, x)$  holds for all  $x, y \in \mathbb{R}$ .

1. Let  $P(x, y) = x^3 + ax^2y + axy^2 + y^3$  be a symmetric polynomial of degree 3. Prove that  $P(x, y) > 0$  for all  $x, y > 0$  if and only if  $a > -1$ .
2. Let  $P(x, y) = x^4 + ax^3y + bx^2y^2 + axy^3 + y^4$  be a symmetric polynomial of degree 4.
- (a) Prove that  $P(x, y) > 0$  for all  $x, y > 0$  if and only if  $a < -4$ ,  $b > \frac{a^2+8}{4}$  or  $a \geq -4$ ,  $b > -2a - 2$ .
- (b) Prove that  $P(x, y) > 0$  for all  $x, y \neq 0$  if and only if  $|a| > 4$ ,  $b > \frac{a^2+8}{4}$  or  $|a| \leq 4$ ,  $b > 2|a| - 2$ .

3. Let  $P(x, y)$  be one of the following symmetric polynomials:

$$\begin{aligned} & x^5 + ax^4y + bx^3y^2 + bx^2y^3 + axy^4 + y^5, \\ & x^6 + ax^5y + bx^4y^2 + cx^3y^3 + bx^2y^4 + axy^5 + y^6, \\ & x^7 + ax^6y + bx^5y^2 + cx^4y^3 + cx^3y^4 + bx^2y^5 + axy^6 + y^7. \end{aligned}$$

Find necessary and sufficient conditions on the coefficients such that  $P(x, y) > 0$  for all (a)  $x, y > 0$ , (b)  $x, y \neq 0$ .

4. Find sufficient conditions for a homogeneous symmetric polynomial  $P(x, y)$  of degree  $n > 7$  to take positive values for all (a)  $x, y > 0$ , (b)  $x, y \neq 0$ .

5. Using methods developed in the previous questions, give necessary and sufficient conditions for a non-homogeneous symmetric polynomial of two real variables to be positive.

### 9. Good Numbers

Any rational number  $x$  may be expressed as a continued fraction

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{\ddots}{\ddots} + \frac{1}{a_m}}}$$

where  $a_0$  is the integer part of  $x$ , and the numbers  $a_1, a_2, \dots, a_m$  are positive integers called *partial quotients* of  $x$ . We will also write  $x = [a_0; a_1, a_2, \dots, a_m]$ .

1. Find all numbers  $n > 2$  that can be expressed as the sum of two positive integers  $n = a + b$  so that  $a < b$  and the continued fraction for  $a/b$  has all its partial quotients equal to 1. For example,  $13 = 5 + 8$  and  $5/8 = [0; 1, 1, 1, 1, 1]$ .

2. A number  $n > 2$  is called *2-good* if for some positive integers  $a < b$  we have  $n = a + b$  and the partial quotients of  $a/b$  are equal to 1 or 2. For example,  $11 = 4 + 7$  and  $4/7 = [0; 1, 1, 2, 1]$ .

(a) Are there infinitely many odd numbers that are *not* 2-good?

(b) Is it true that any even positive integer, greater than 6, is the sum of two distinct odd 2-good numbers? If it is not, find all even numbers with this property.

(c) Describe the set of all 2-good numbers.

3. In general, a number  $n$  is called *k-good* if it can be expressed as the sum of two positive integers  $n = a + b$  so that  $a < b$  and the continued fraction for  $a/b$  has all its partial quotients not greater than  $k$ .

Does there exist a positive integer  $K$  such that all positive integers  $n > 2$  are  $K$ -good?

4. Describe the set of numbers  $n$  with the property: there exist two positive integers  $b > a$  such that  $n = b + a$  or  $n = b - a$  and the partial quotients of  $a/b$  are equal to 1 or 2.

5. Suggest and study additional directions of research.

### 10. Centro-Symmetric Shadows

A set of points in the plane is said to be *centro-symmetric* if it has a centre of symmetry. For example, the vertices of a square form a centro-symmetric set. (Recall that a *centre of*

*symmetry* of a set  $S$  in the plane is a point  $c$  with the property that for any point  $p \in S$  there exists a point  $p' \in S$  such that  $p$  and  $p'$  are equidistant from  $c$  and lie on a line passing through  $c$ .)

Given a set of points, its *shadow* on a line is its orthogonal projection onto this line.

1. Let  $n \geq 3$  be a positive integer. Denote by  $k(n)$  the minimum positive integer  $k$  with the following property:

for any set  $S$  of  $n$  points in the plane, if there exist  $k$  lines, no two parallel, such that for each line the shadow of  $S$  on this line is centro-symmetric, then the initial set  $S$  is also centro-symmetric.

Find or estimate the number  $k(n)$ .

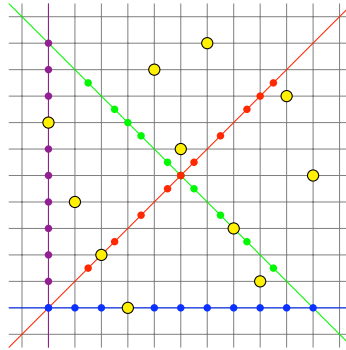


FIGURE 5. A set of 11 points with 4 centro-symmetric shadows:  $k(11) > 4$ .

2. Formulate and study 3-dimensional analogs of the problem.

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